# Competing Brownian Particles

Andrey Sarantsev

A dissertation
submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2015

Reading Committee:

Soumik Pal, Chair

Krzysztof Burdzy

Zhen-Qing Chen

Program Authorized to Offer Degree:
University of Washington, Seattle
Department of Mathematics

University of Washington

**Abstract**

Competing Brownian Particles

Andrey Sarantsev

Chair of the Supervisory Committee:
Associate Professor Soumik Pal
Department of Mathematics

Consider a finite system of $N$ Brownian particles on the real line. Rank them from bottom to top: the (currently) lowest particle has rank 1, the second lowest has rank 2, etc., up to the top particle, which has rank $N$. The particle which has (currently) rank $k$ moves as a Brownian motion with drift coefficient $g_k$ and diffusion coefficient $\sigma_k^2$. When two or more particles collide, they might exchange ranks; in this case, they exchange drift and diffusion coefficients. This model is called a system of competing Brownian particles. It was introduced in Banner, Fernholz, Karatzas (2005) for the purpose of financial modeling. Since then, it attracted a considerable amount of attention.

We can also consider infinite systems of competing Brownian particles (with the lowest particle but no highest particle, that is, with ranks ranging from 1 to $\infty$). For both finite and infinite systems, the gap process is formed by the spacings (gaps) between adjacent particles. It is $N-1$-dimensional for a finite system with $N$ particles and infinite-dimensional for an infinite system. We say that a triple collision has occurred if three or more particles occupy the same position at the same time.

In this thesis, we prove several new results about these systems. In particular, we establish convergence results for the gap process of infinite systems, building on the work of Pal, Pitman (2008); and we find a necessary and sufficient condition for a.s. absence of triple collisions, continuing the research from Ichiba, Karatzas, Shkolnikov (2013).

# TABLE OF CONTENTS

# ACKNOWLEDGMENTS

First and foremost, I wish to express foremost gratitude to my Ph.D. adviser, Soumik Pal. He was guiding me through the arduous task of finding research problems, reading relevant literature, proving new results, writing them in a publishable form, and going through the painstaking publication process. He is a fantastic adviser - an overused phrase which really applies here. Even when I was completely at a loss, he did not lose his faith in me and continued to support me. Without this, I would not be able to do anything. Thank you, Professor Pal!

Next, I would like to thank other faculty who study Probability: Krzysztof Burdzy, Zhen-Qing Chen, Ioana Dumitriu, Christopher Hoffman, Steffen Rohde. Discussions with them about Probability and beyond helped a lot. I took some of their Special Topics courses, and this greatly enriched my basic knowledge about different areas of Probability. This feeling of vibrant Probability community in the Department of Mathematics is fantastic! It seems that this is one of the strongest departments in the U.S. with regard to Probability. I was really lucky to complete my Ph.D. in this place.

I would also like to deeply thank Ioannis Karatzas, a professor at Columbia University and Soumik Pal's former Ph.D. adviser. Ioannis' support was crucial for finding a research topic. His monthly meetings at INTECH headquarters in Princeton exposed me to the whole community of Ioannis' current and former Ph.D. students, as well as other scholars, who work in similar areas. This opened the door to the whole new world. And his full financial support for those meetings made a big difference at the time when I was living on a salary of a graduate student. Thanks, Professor Karatzas!

Thanks to Cameron Bruggeman, a (current) Ph.D. student of Ioannis Karatzas. Collaboration with him is fantastic. At a few occasions, his insights broke the wall and provided precisely the ideas we needed to move forward with our research.

Thanks to Mykhaylo Shkolnikov, an assistant professor at the University of Maryland, College Park. Conversations with him during INTECH meetings and during my visits to Berkeley and Stanford helped me move forward. Much of what I did in this thesis is a continuation of Misha's work.

Thanks to Tomoyuki Ichiba and Jean-Pierre Fouque, who invited me in March 2014 to give a talk at UC Santa Barbara on the Financial Mathematics Seminar. They wecolmed me very warmly and turned this visit into a fantastic experience. Now, Jean-Pierre Fouque has kindly invited me to a three-year postdoctoral position. Also, Tomoyuki suggested a problem which lead to my first paper (different from the ones included in this thesis).

Thanks to Amir Dembo and Vladas Sidoravicius for inviting me to Stanford in April 2014. Thanks to Manuel Cabezas, a postdoc of Vladas, for useful discussion during my visit to Stanford.

Thanks to Ruth Williams from UC San Diego, for numerous very helpful discussions about reflected Brownian motion.

Thanks to Adrian Banner, Robert Fernholz, Philip Whitman, and Vassilios Papathanakos, for their discussions and warm welcome at INTECH meetings, as well as at the Stochastic Portfolio Theory conference at Columbia University in May 2015. Thanks to Yinghui Wang, a Ph.D. student of Ioannis Karatzas, for organizing these INTECH meetings.

Thanks to Boris Solomyak, who helped me during my first quarter here with respect to teaching, when I was getting used to a totally different system of education than in Russia. He is from Russia, too, so he understood the differences between the teaching systems and the challenges I was facing at the time.

Thanks to Judith Arms and Andrew Loveless for coaching me how to teach classes. They helped me a lot to improve teaching skills: presentation, organizing materials, blackboard skills, giving fair grades, efficiently calculating final scores and grades, etc. They are real experts in undergraduate teaching, and they very kindly passed some of their experience to me, so that I did not have to learn this from scratch by trial and error.

Thanks to Microsoft Research for organizing a fantastic one-day Northwest Probability Seminar, which gathers probability people from the west side of Lake Washington (University of Washington) and the east side of this lake (Microsoft Research). Seattle is truly a great place to learn probability!

Thanks to Douglas Rizzolo and Brent Werness, NSF postdocs at the Department of Mathematics who studied Probability. It was nice chatting with them about Probability and other stuff.

Thanks to Douglas Lind, John Palmieri, Brooke Miller, Chris Bonneu, Michael Munz, Garrett Yoshitomi, and Mary Sheetz for providing organizational support during these five years.

Thanks to Marina Meila-Predoviciu and Adrian Dobra, who work in the Department of Statistics. Their courses were interesting, and I enjoyed them, although, frankly, I didn't do very well in them.

Thanks to my professors at Lomonosov Moscow State University, in particular Vladimir Piterbarg (my undergraduate adviser), Alexey Shashkin, Katherine Bulinskaya and Andrey Selivanov. They provided me with great education in Probability that proved to be a good foundation for my Ph.D. studies.

Thanks to my teachers at 57th high school, a top math high school in Moscow, Russia.

Thanks to Robert Torney, a landlord who provided fantastic housing in Seattle for me. His positive atittude and comfort which he created in his house really made a difference when I was immersed into this daunting task of writing Ph.D. thesis.

Thanks to my parents: my mother Marina, and my father Andrey, who taught me the virtues of learning and hard work. Without them, I would not be able to accomplish this. Thanks also to my aunt Anna and uncle Maksim, as well as to my grandparents Valentina and Mikhail for their support.

Thanks to my friend Yury Dvorkin for useful discussion.

Last but not least, thanks to all math graduate students, who I was fortunate to meet in this Ph.D. program. In particular, I would like to thank the following (current and former) Ph.D. students who are studying Probability: Tvrtko Tadic, Clayton Barnes, Sudip Paul, Sayan Banerjee, Tobias Johnson, Elliot Paquette, Wai-Tong Fan, Tran vo Huy, Shuwen Lou, Christian Rudnick, Erik Slivken, Matthew Junge, Ting-Kam Wong, Jiashan Wang, Harishchandra Ramadas, Shirshendu Ganguly, Avi Levy, Gerandy Brito Mondes de Oca, Peter Lin, Joel Barnes, Nathaniel Blair-Stahn, Mauricio Duarte, Joshua Tokie.

Also, I would like to thank my other friends: Timothy Mesikepp, Bharathwaj Palvannan, Kyle Curlett, Stephen McKeown, Stephen Lewis, Jessica Merhej, Cris Negron, Abdalia Nimer, Peter Caday, Yernat Assylbekov, Alan Bartlett, Lindsay Erickson, Alys Lim, Graham Clenaghan, Gabriel Dorfsman-Hopkins, Kristin DeVleming, Rebecca Hoberg, Rebecca Uhlman, Courtney Kempton, Lorenzo Prelli, David Sprehn, James Stark, James Pfeiffer, and other math graduate students. It was great joy and fun to be in this company.

This work was partially supported by NSF grants DMS 1007563, DMS 1308340, and DMS 1405210.

# DEDICATION

to my father

and to the memory of my mother

# NOTATION

We denote by $I_k$ the $k \times k$-identity matrix. We let $\mathbb{R}_+ := [0, \infty)$. For a vector $x = (x_1, \ldots, x_d)' \in \mathbb{R}^d$, let $\|x\| := (x_1^2 + \ldots + x_d^2)^{1/2}$ be its Euclidean norm. For any two vectors $x, y \in \mathbb{R}^d$, their *dot product* is denoted by $x \cdot y = x_1 y_1 + \ldots + x_d y_d$. We compare vectors $x$ and $y$ componentwise: $x \le y$ if $x_i \le y_i$ for all $i = 1, \ldots, d$; $x < y$ if $x_i < y_i$ for all $i = 1, \ldots, d$; similarly for $x \ge y$ and $x > y$. This comparison notation is also valid when $d = \infty$, that is, when we compare infinite-dimensional vectors. We compare matrices of the same size componentwise, too. For example, we write $x \ge 0$ for $x \in \mathbb{R}^d$ if $x_i \ge 0$ for $i = 1, \ldots, d$; $C = (c_{ij})_{1 \le i,j \le d} \ge 0$ if $c_{ij} \ge 0$ for all $i$, $j$. The symbol $a'$ denotes the transpose of (a vector or a matrix) $a$.

Fix $d \ge 1$, and let $I \subseteq \{1, \ldots, d\}$ be a nonempty subset. Write its elements in increasing order: $I = \{i_1, \ldots, i_m\}$, $1 \le i_1 < i_2 < \ldots < i_m \le d$. For any $x \in \mathbb{R}^d$, let $[x]_I := (x_{i_1}, \ldots, x_{i_m})'$. For any $d \times d$-matrix $C = (c_{ij})_{1 \le i,j \le d}$, let $[C]_I := \left(c_{i_k i_l}\right)_{1 \le k,l \le m}$. Let $J \subseteq \{1, \ldots, d\}$ be another nonempty subset. Write its elements in the order of increase:

$$J = \{j_1, \ldots, j_l\}, \quad 1 \le j_1 < j_2 < \ldots < j_l \le d.$$

Then we denote

$$[C]_{IJ} := (c_{i_k j_s})_{\substack{1 \le k \le m \\ 1 \le s \le l}}.$$

In particular, for $I = J$ we have: $[C]_{IJ} \equiv [C]_I$. If $p = 1, \ldots, d$, then we let $[x]_p := (x_1, \ldots, x_p)'$. We let

$$\mathcal{W}_N := \{y = (y_1, \ldots, y_N)' \in \mathbb{R}^N \mid y_1 \le \ldots \le y_N\}.$$

We write $C([0,T], \mathbb{R}^d)$ for the set of continuous functions $f : [0,T] \to \mathbb{R}^d$. For the case $d = 1$, we write $C[0,T] \equiv C([0,T], \mathbb{R}^d)$. For $A \subseteq \mathbb{R}^d$, we write $C_b^2(A)$ for the set of twice continuously differentiable functions $f : A \to \mathbb{R}$ which are bounded together with their first and second derivatives.

For $x \in \mathbb{R}^d$ (this includes the case $d = \infty$), we let $[x, \infty) := \{y \in \mathbb{R}^d \mid y \ge x\}$. We say two probability measures $\nu_1$ and $\nu_2$ on $\mathbb{R}^d$ satisfy $\nu_1 \preceq \nu_2$, or $\nu_2 \succeq \nu_1$, if for every $y \in \mathbb{R}^d$ we have: $\nu_1[y, \infty) \le \nu_2[y, \infty)$. We say that $\nu_1$ *is stochastically dominated by* $\nu_2$, or $\nu_2$ *stochastically dominates* $\nu_1$, or $\nu_1$ *is stochastically smaller than* $\nu_2$, or $\nu_2$ *is stochastically larger than* $\nu_1$. The same terminology applies to $\mathbb{R}^d$-valued random variables $X$, $Y$: we say that $X$ is stochastically dominated by $Y$ if the distribution of $X$ is stochastically dominated by the distribution of $Y$.

Consider two $\mathbb{R}^d$-valued processes $Z = (Z(t), t \ge 0)$ and $\overline{Z} = (\overline{Z}(t), t \ge 0)$. (This includes the case $d = \infty$.) We say that $Z$ is *stochastically dominated by* $\overline{Z}$, and write it as $Z \preceq \overline{Z}$, if for every $t \ge 0$ and $y \in \mathbb{R}^d$ we have:

$$\mathbf{P}(Z(t) \ge y) \le \mathbf{P}(\overline{Z}(t) \ge y).$$

In otehr words, $Z \preceq \overline{Z}$ if for every $t \ge 0$ we have: $Z(t) \preceq \overline{Z}(t)$. If the processes $Z$ and $\overline{Z}$ are Markov, then by changing the probability space we can move from stochastic domination to pathwise domination, see [70, Theorem 5].

The arrow $\Rightarrow$ indicates weak convergence of probability measures or random variables. The symbol $\mathcal{E}(\alpha)$ stands for the exponential distribution with mean $\alpha^{-1}$, rate $\alpha$ and density $\alpha e^{-\alpha x}\mathrm{d}x,\ x > 0$. A *standard Brownian motion* is a one-dimensional Brownian motion starting from zero with drift coefficient 0 and diffusion coefficient 1. The symbol $\delta_x$ indicates the Dirac delta measure at the point $x$.

We call the sequence $(a_1, \ldots, a_n)$ of real numbers *concave* if

$$a_k \ge \frac{1}{2}\left(a_{k+1} + a_{k-1}\right), \quad k = 2, \ldots, n-1.$$

Same definition applies to a sequence $(a_1, a_2, \ldots)$.

For a positive multidimensional orthant $S = \mathbb{R}^d_+$, we let $S_i = \{x \in S \mid x_i = 0\}$ be the $i$th face of the boundary $\partial S$. For a nonempty subset $I \subseteq \{1, \ldots, d\}$, we let

$$S_I := \{x \in S \mid x_i = 0 \text{ for } i \in I\}.$$

This is called an *edge* of the boundary $\partial S$ of the orthant $S$.

We use the following metric on $\mathbb{R}^\infty$:

$$\rho\left((x_n)_{n\geq 1},(y_n)_{n\geq 1}\right):=\sum_{n=1}^{\infty}2^{-n}\left(1\wedge|x_n-y_n|\right). \tag{1}$$

This metric corresponds to the componentwise convergence.

We denote by

$$\Psi(u):=\frac{1}{\sqrt{2\pi}}\int_u^{\infty}e^{-x^2/2}\mathrm{d}x$$

the tail probability of the standard normal distribution.

# Chapter 1

# INTRODUCTION

## *1.1 The Concept of Competing Brownian Particles*

Consider a system $X(t) = (X_1(t), \ldots, X_N(t))'$ of $N$ Brownian particles on the real line. Rank them from bottom to top:

$$X_{(1)}(t) \le X_{(2)}(t) \le \ldots \le X_{(N)}(t).$$

The (currently) lowest particle has rank 1, the next lowest particle has rank 2, etc., up to the top particle, which has rank $N$. Assume they move according to the following rule: the particle which currently has rank $k$ moves as a Brownian motion with drift coefficient $g_k$ and diffusion coefficient $\sigma_k^2$, for each $k = 1, \ldots, N$. Particles can collide and exchange ranks; in this case, they exchange their drift and diffusion coefficients.

(There is a technical difficulty: if a few particles occupy the same position, then how do we assign ranks? We *resolve ties in favor of the lexicographic order*; more on that later.)

In other words, the SDE governing this system is

$$\mathrm{d}X_i(t) = \sum_{k=1}^{N} 1\left(X_i \text{has rank } \ k \ \text{ at time } \ t\right)\left(g_k \mathrm{d}t + \sigma_k \mathrm{d}W_i(t)\right), \tag{1.1}$$

where $W_1, \ldots, W_N$ are i.i.d. standard Brownian motions. In is proved in [6] that this SDE has a weak solution which is unique in law.

This system was introduced in [3]. The original motivation to study systems of competing Brownian particles came from Stochastic Finance. An observed phenomenon of real-world stock markets is that stocks with smaller capitalizations have larger growth rates and larger volatilities. This can be captured by a system of competing Brownian particles: just let $g_1 > \ldots > g_N$ and $\sigma_1 > \ldots > \sigma_N$, and suppose that for $i = 1, \ldots, N$, the quantity $e^{X_i(t)}$ is the capitalization of the $i$th stock at time $t$. For financial applications and market models similar to this rank-based model, see the articles [2], [29], [72], the book [27, Chapter 5] and a somewhat more recent survey [31, Chapter 3].

This model was recently studied in [59], [58], [30], [11], and other papers. A more extensive literature review can be found in Chapter 3, Section 3.9.

A particular case is the *Atlas model*, when

$$g_1 = 1, \quad g_2 = \ldots = g_N = 0, \quad \sigma_1 = \ldots = \sigma_N = 1.$$

There, the (currently) bottom particle moves as a Brownian motion with drift 1, all other particles move as standard Brownian motions.

The *gap process* is defined as an $\mathbb{R}_+^{N-1}$-valued process $Z(t) = (Z_1(t), \ldots, Z_{N-1}(t))'$, with

$$Z_k(t) = X_{(k+1)}(t) - X_k(t), \quad k = 1, \ldots, N-1, \quad t \geq 0.$$

## 1.2 Collisions of Particles

A *triple collision* occurs when we have:

$$X_{(k-1)}(t) = X_{(k)}(t) = X_{(k+1)}(t) \ \text{ for some } \ k = 2, \ldots, N-1, \quad t > 0.$$

A *simultaneous collision* occurs when we have:

$$X_{(k)}(t) = X_{(k+1)}(t) \ \text{ and } \ X_{(l)}(t) = X_{(l+1)}(t) \ \text{ for some } \ 1 \leq k < l \leq N-1 \ \text{ and } \ t > 0.$$

A triple collision is a particular case of a simultaneous collision (let $k = l - 1$).

One motivation to study triple collisions is that the equation (1.1) has a strong solution (which is pathwise unique) only up to the first moment of a triple collision. After this moment, it is not known whether it has a strong solution or not. (This is the result from [59].)

In the paper [59], it was proved that if the sequence $(0, \sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2, 0)$ is concave, then there are a.s. no triple collisions. Conversely, if there are a.s. no triple collisions, then the sequence $(\sigma_1^2, \ldots, \sigma_N^2)$ is concave. In Chapter 4, we prove the following result:

**Theorem 1.2.1.** *There are a.s. no triple and no simultaneous collisions if and only if the sequence* $(\sigma_1^2, \ldots, \sigma_N^2)$ *is concave. If it is not concave, that is, there exists* $k = 2, \ldots, N-1$ *such that*

$$\sigma_k^2 < \frac{1}{2}\left(\sigma_{k-1}^2 + \sigma_{k+1}^2\right),$$

*then with positive probability there is a triple collision between particles* $X_{(k-1)}, X_{(k)}, X_{(k+1)}$.

An interesting corollary: if there are a.s. no triple collisions, then there are a.s. no simultaneous collisions.

We also find sufficient conditions for absence of *specific types of collisions*: for example, $X_{(1)}(t) = X_{(2)}(t) = X_{(3)}(t)$. Suppose, for the sake of example, that we have $N = 4$ competing Brownian particles.

**Proposition 1.2.2.** *If the following conditions*

$$\begin{cases} 9\sigma_1^2 \le 7\sigma_2^2 + 7\sigma_3^2 + 7\sigma_4^2; \\ 3\sigma_1^2 \le 5\sigma_2^2 + \sigma_3^2 + \sigma_4^2; \\ 3\sigma_1^2 + 3\sigma_4^2 \le 5\sigma_2^2 + 5\sigma_3^2; \\ 3\sigma_4^2 \le \sigma_1^2 + \sigma_2^2 + 5\sigma_3^2; \\ 9\sigma_4^2 \le 7\sigma_1^2 + 7\sigma_2^2 + 7\sigma_3^2, \end{cases}$$

*hold, then a.s. there does not exist* $t > 0$ *such that*

$$X_{(1)}(t) = X_{(2)}(t) = X_{(3)}(t) = X_{(4)}(t). \tag{1.2}$$

Moreover, we can make a stronger statement.

**Proposition 1.2.3.** *If the five inequalities from Proposition* 1.2.2 *hold, then a.s. there does not exist* $t > 0$ *such that*

$$X_{(1)}(t) = X_{(2)}(t) \quad \text{and} \quad X_{(3)}(t) = X_{(4)}(t). \tag{1.3}$$

**Proposition 1.2.4.** *If the five inequalities from Proposition* 1.2.2 *together with*

$$\sigma_2^2 \ge \frac{1}{2}\left(\sigma_1^2 + \sigma_3^2\right)$$

*hold, then a.s. there does not exist* $t > 0$ *such that*

$$X_{(1)}(t) = X_{(2)}(t) = X_{(3)}(t). \tag{1.4}$$

Similar statements (but with other inequalities involving $\sigma_k^2,\ k = 1, \ldots, N$) can be stated for any $N = 5, 6, \ldots$ and for any type of collision between $X_{(1)}, \ldots, X_{(N)}$ (for example, $X_{(2)} = X_{(3)} = X_{(4)}$ and $X_{(6)} = X_{(7)}$).

It is also shown in Theorem 6.2.2 due to Cameron Bruggeman that in the $N = 4$ case above, if $\sigma_1^2 + \sigma_4^2 \le \sigma_2^2 + \sigma_3^2$, then there a.s. there is no $t > 0$ such that (1.2) holds. However, this result has no generalizations for larger $N$.

### *1.3 Sketch of Proof for Collisions: an SRBM in the Orthant*

The idea of the proof is as follows. The gap process is the so-called *semimartingale reflected Brownian motion*, shortly SRBM, in the orthant $\mathbb{R}_+^{N-1}$. We devote Chapter 2 to this process; now let us just explain it informally. See also the papers [125], [124] for the background. An extensive literature review is postopned until Section 2.5.

Fix $d \ge 2$, the dimension. Let $S = \mathbb{R}_+^d$ be the $d$-dimensional positive orthant. Let us loosely describe an $S$-valued stochastic process $Z = (Z(t), t \ge 0)$, which is called a *semimartingale reflected Brownian motion*, shortly SRBM, *in the orthant* $S$. First, let us describe its parameters: a *drift vector* $\mu \in \mathbb{R}^d$, a symmetric positive definite $d \times d$ *covariance matrix* $A$, and another $d \times d$ *reflection matrix* $R$. The process $Z = (Z(t), t \ge 0)$, which is also denoted by $\mathrm{SRBM}^d(R, \mu, A)$, has the following properties:

(i) it behaves as a $d$-dimensional Brownian motion with drift vector $\mu$ and covariance matrix $A$ in the interior of $S$;

(ii) for each $i = 1, \ldots, d$, at the *face* $S_i = \{z \in S \mid z_i = 0\}$ of the boundary $\partial S$, it is reflected according to the direction $r_i$, the $i$th column of $R$.

The gap process is an $\mathrm{SRBM}^{N-1}(R, \mu, A)$ with the following parameters $R, \mu, A$:

$$R = \begin{bmatrix} 1 & -1/2 & 0 & \dots & 0 \\ -1/2 & 1 & -1/2 & \dots & 0 \\ 0 & -1/2 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} \sigma_1^2 + \sigma_2^2 & -\sigma_2^2 & 0 & \dots & 0 \\ -\sigma_2^2 & \sigma_2^2 + \sigma_3^2 & -\sigma_3^2 & \dots & 0 \\ 0 & -\sigma_3^2 & \sigma_3^2 + \sigma_4^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sigma_{N-1}^2 + \sigma_N^2 \end{bmatrix}$$

$$\mu = (g_2 - g_1, \dots, g_N - g_{N-1})'.$$

From the definition of the gap process $Z$, one can immediately see that there a.s. no triple and simultaneous collisions if and only if the gap process $Z$ *does not hit non-smooth parts of the boundary* (intersections of two or more faces):

$$\bigcup_{1 \le i < j \le d} (S_i \cap S_j).$$

The main result of Chapter 4, which corresponds to the author's paper [103], is as follows. Under some additional technical conditions, an $\mathrm{SRBM}^d(R, \mu, A)$ a.s. does not hit non-smooth parts of the boundary if and only if

$$RD + DR' \ge 2A, \tag{1.5}$$

where $D = \mathrm{diag}(A)$ is a $d \times d$-diagonal matrix with the same diagonal entries as $A$. Then we translate this condition into the language of $\sigma_1^2, \dots, \sigma_N^2$.

Similar method applies to particular types of collisions, as in Examples 1.2.2, 1.2.3, 1.2.4. Then, we find sufficient condition when an $\mathrm{SRBM}^d(R, \mu, A)$ does not hit a particular edge of the boundary $\partial S$ of the orthant, say $\{z \in S \mid z_1 = z_2 = 0\} = S_1 \cap S_2$. (This particular edge corresponds to the collision $X_{(1)}(t) = X_{(2)}(t) = X_{(3)}(t)$.)

### *1.4 Infinite Atlas Model*

Consider now infinite systems of competing Brownian particles, in particular, the *infinite Atlas model*. They are defined in the same way as finite systems, only the ranks of the particles go from 1 to $\infty$, from bottomt to top. So there is the lowest particle at every moment, but no highest particle. The gap process is defined similarly; now it is an $\mathbb{R}^{\infty}_{+}$-valued process.

Existence and uniqueness of such models (even the weak one) is much harder to establish than that for finite systems: see, for example, [105], [89], and [59]. We find some new results about existence and uniqueness in Chapter 7 of this thesis.

In the paper [89], it was shown that for the Atlas model with $N$ particles, the gap process has the following stationary distribution:

$$\pi^{(N)} := \bigotimes_{k=1}^{N-1}\mathcal{E}\left(2\frac{N-k}{N}\right). \tag{1.6}$$

Also, it was proved in [89] that for the infinite Atlas model, the gap process has the following stationary distribution:

$$\pi_{\infty} := \bigotimes_{k=1}^{\infty}\mathcal{E}(2). \tag{1.7}$$

The idea of the proof is as follows. We approximate the infinite Atlas model by finite Atlas models. Fsor every $k \ge 1$, we have:

$$2\frac{N-k}{N} \to 2 \quad \text{as} \quad N \to \infty,$$

and so, in some sense, $\pi^{(N)}$ "tends" to $\pi_{\infty}$.

In this thesis, we prove the following results. (The convergence in $\mathbb{R}^{\infty}$ is with respect to the metric $\rho$, that is, componentwise.)

**Theorem 1.4.1.** *(i) For any copy of the the infinite Atlas model, the family of random variables $(Z(t),\ t \ge 0)$ is tight in $\mathbb{R}^{\infty}_{+}$ with respect to the metric $\rho$. Moreover, any weak limit point of $Z(t)$ as $t \to \infty$ is stochastically dominated by $\pi_{\infty}$. In other words, if $(t_j)_{j\ge1}$ is an* *increasing sequence of positive time moments such that*

$$t_j \to \infty \quad \text{and} \quad Z(t_j) \Rightarrow \nu, \quad \text{then} \quad \nu \preceq \pi_\infty.$$

*(ii) Let $\nu$ be a probability measure on $\mathbb{R}_+^\infty$ such that $\pi_\infty \preceq \nu$. Start the infinite Atlas model with $Z(0) \backsim \nu$. Such model exists in the strong sense and is pathwise unique, and*

$$Z(t) \Rightarrow \pi_\infty, \quad t \to \infty.$$

*(iii) Consider an infinite Atlas model such that for some probability measure $\nu$ on $\mathbb{R}_+^\infty$ we have: $Z(t) \backsim \nu$ for all $t \geq 0$. Then $\nu \preceq \pi_\infty$.*

In other words, any limit point of the gap process has gaps stochastically smaller than $\pi_\infty$. Moreover, if we start the infinite Atlas model with gaps stochastically larger than $\pi_\infty$, then the gaps will converge to $\pi_\infty$. However, in other cases (for example, when initially the gaps are stochastically smaller than $\pi_\infty$), we do not know whether the weak limit exists.

### 1.5 Sketch of Proof for the Infinite Atlas Model

We approximate the infinite Atlas model $X$ by a finite Atlas model of $N$ particles

$$X^{(N)} = \left(X_1^{(N)}, \ldots, X_N^{(N)}\right)'.$$

Rank these particles:

$$X_{(1)}^{(N)}(t) \leq X_{(2)}^{(N)}(t) \leq \ldots \leq X_{(N)}^{(N)}(t),$$

Then, as mentioned before in (1.6), the gap process for the finite Atlas model $X^{(N)}$:

$$Z^{(N)} = (Z^{(N)}(t), t \geq 0), \quad Z^{(N)}(t) = (Z_1^{(N)}(t), \ldots, Z_N^{(N)}(t))',$$

defined by

$$Z_k^{(N)}(t) = X_{(k+1)}^{(N)}(t) - X_{(k)}^{(N)}(t), \quad k = 1, \ldots, N-1, \quad t \geq 0,$$

has a unique stationary distribution

$$\pi^{(N)} := \bigotimes_{k=1}^{N-1} \mathcal{E}\left(2\frac{N-k}{N}\right),$$

and converges weakly to this distribution as $t \to \infty$, regardless of the initial distribution. Suppose that the initial conditions for ranked particles are the same:

$$X_{(k)}^{(N)}(0) = X_{(k)}(0), \quad k = 1, \ldots, N.$$

You can turn an infinite Atlas model into a finite Atlas model with $N$ particles by removing the $(N+1)$st, $(N+2)$nd, etc. ranked particles $X_{(N+1)}, X_{(N+2)}, \ldots$ When two particles $X_{(N)}$ and $X_{(N+1)}$ in the infinite Atlas model collide, they are pushed apart (because the model stipulates that $X_{(N)}(t) \le X_{(N+1)}(t)$). The particle $X_{(N)}$ "feels the pressure" from $X_{(N+1)}$ from above.

Now comes the crucial step. Here, we use the comparison statement (proved in [100, Corollary 3.9], see also Corollary 4.3.8 in Chapter 4 of this thesis). This "pressure from above" decreases the gaps

$$Z_1 = X_{(2)} - X_{(1)}, \ldots, Z_{N-1} = X_{(N)} - X_{(N-1)}$$

between adjacent particles $X_{(1)}, \ldots, X_{(N)}$. In other words, these gaps are smaller in the infinite Atlas model than they would be without $X_{(N+1)}, X_{(N+2)}, \ldots$, that is, than they would be in the finite Atlas model with $N$ particles. We can write this as

$$Z_k(t) \le Z_k^{(N)}(t), \quad k = 1, \ldots, N-1, \; t \ge 0. \tag{1.8}$$

But

$$Z^{(N)}(t) = \left(Z_1^{(N)}(t), \ldots, Z_N^{(N)}(t)\right)' \Rightarrow \pi^{(N)}, \quad t \to \infty.$$

So for $k = 1, \ldots, N-1$,

$$Z_k^{(N)}(t) \Rightarrow \mathcal{E}\left(2\frac{N-k}{N}\right), \quad t \to \infty. \tag{1.9}$$

Combining (1.8) and (1.9), we get: for every component $Z_k(t)$ of the gap process, every weak limit point is stochastically dominated by $\mathcal{E}(2(N-k)/N)$ for any $N > k$. But $N$ is arbitrary, and

$$2\frac{N-k}{N} \to 2, \quad N \to \infty,$$

for any fixed $k$. So any weak limit point of $Z_k(t)$ is stochastically dominated by $\mathcal{E}(2)$, for each $k = 1, 2, \ldots$ Slightly changing the argument, we can show a stronger statement: any weak limit point of $Z(t)$ is stochastically dominated by

$$\pi_\infty := \bigotimes_{k=1}^{\infty} \mathcal{E}(2).$$

This proves part (i) of Theorem 1.4.1. Let us prove part (ii). Suppose we start the infinite Atlas model $X$ with the gaps stochastically larger than $\pi_\infty$: $Z(0) \succeq \pi_\infty$. The gap process is *stochastically ordered*: if we start from (stochastically) larger gaps, then for every $t \ge 0$ the gaps will also be stochasticallfy larger. For the finite Atlas model, this follows from Corollary 4.3.10(ii) from Chapter 4, which corresponds to [100, Corollary 3.11(ii)]. For the infinite Atlas model, we show this fact in this chapter, in Corollary 7.3.6, using approximation by finite Atlas models.

As mentioned earlier, the distribution $\pi_\infty$ is a stationary distribution for the gap process of the infinite Atlas model. In other words, suppose we start a copy $\overline{X}$ of the infinite Atlas model with the corresponding gap process $\overline{Z}$ initially distributed as $\pi_\infty$: $\overline{Z}(0) \backsim \pi_\infty$. Then $\overline{Z}(t) \backsim \pi_\infty$ for all $t \ge 0$. But $Z(0) \succeq \pi_0 \backsim \overline{Z}(0)$, so by the stochastic ordering of $Z$ we conclude: $Z(t) \succeq \overline{Z}(t) \backsim \pi_\infty$ for all $t \ge 0$. On the other hand, any weak limit point of $Z$, as we have just shown, must be stochastically smaller than $\pi_\infty$. It follows that $Z(t) \Rightarrow \pi_\infty$ as $t \to \infty$.

Finally, the part (iii) of Theorem 1.4.1 follows directly from part (i).

### *1.6 Organization of the Thesis*

Chapter 2 provides background and states already known facts about an SRBM in the orthant. Chapter 3 does the same for systems of competing Brownian particles. These two chapters contain almost exclusively the results which are already known. The next four chapters correspond to the four papers written by the author (which constitute the core of thesis):

- Chapter 4, which corresponds to the paper [100], develops comparison techniques, on which the subsequent proofs are based; the reader has seen some of these techniques in the sketch of proof of Theorem 1.4.1.

- Chapter 5, which corresponds to the paper [103], proves the condition (1.5), as well as Theorem 1.2.1 (as a corollary);

- Chapter 6, which corresponds to the paper [102], deals with statements like Examples 1.2.2, 1.2.3, 1.2.4. We prove a more general result and get these examples as corollaries;

- Chapter 7, which corresponds to the paper [101], proves results about infinite systems, including Theorem 1.4.1. We also added proofs of already known statements from [105] and [59] to this chapter for the sake of completeness.

Chapter 8 is devoted to some related infinite systems of Brownian particles: double-sided infinite systems and systems with nonlinear drifts. This material was not included in the paper [101].

# Chapter 2

## SEMIMARTINGALE REFLECTED BROWNIAN MOTION (SRBM) IN THE ORTHANT

Fix $d \ge 2$, the dimension. Let $S = \mathbb{R}^d_+$ be the $d$-dimensional positive orthant. Let us loosely describe an $S$-valued stochastic process $Z = (Z(t), t \ge 0)$, which is called a *semimartingale reflected Brownian motion*, shortly SRBM, *in the orthant* $S$. First, let us describe its parameters: a *drift vector* $\mu \in \mathbb{R}^d$, a symmetric positive definite $d \times d$ *covariance matrix* $A$, and another $d \times d$ *reflection matrix* $R$. The process $Z = (Z(t), t \ge 0)$, which is also denoted by $\mathrm{SRBM}^d(R, \mu, A)$, has the following properties:

(i) it behaves as a $d$-dimensional Brownian motion with drift vector $\mu$ and covariance matrix $A$ in the interior of $S$;

(ii) for each $i = 1, \dots, d$, at the *face* $S_i = \{z \in S \mid z_i = 0\}$, it is reflected according to the direction $r_i$, the $i$th column of $R$.

If $r_i = e_i$, which is the $i$th standard unit vector in $\mathbb{R}^d$, then the reflection on the face $S_i$ is called *normal*. Otherwise, it is called *oblique*.

First, we describe the one-dimensional version of this process: a *reflected Brownian motion on the positive half-line* $\mathbb{R}_+$, which was introduced in the articles [107], [108] by A. V. Skorohod. Then we introduce the deterministic version of this process: the *Skorohod problem*, and use it to define an SRBM in the orthant. Later, we list some relevant properties of this process (without proofs) and conduct a literature review.

### *2.1 Reflected Brownian Motion on the Half-Line*

Consider a Brownian motion $B = (B(t), t \ge 0)$ in dimension one, with zero drift and unit diffusion. Then the process

$$Z = (Z(t), t \ge 0), \quad Z(t) := |B(t)|, \quad t \ge 0,$$

is called a *reflected Brownian motion on* $\mathbb{R}_+$. It behaves as a Brownian motion as long as it stays *inside* the half-line, that is, in $(0, \infty)$. When it hits zero, it is *reflected* in the positive direction, so that it cannot deviate to the negative half-line. Another way to represent the process $Z$ is as follows:

$$Z(t) = W(t) + L(t), \tag{2.1}$$

where

$$W(t) = Z(0) + \int_0^t \mathrm{sign}(B(s))\mathrm{d}B(s), \quad t \ge 0$$

is another version of one-dimensional Brownian motion, and $L = (L(t), t \ge 0)$ is a nondecreasing process with $L(0) = 0$, which can increase only when $Z(t) = 0$. We can write the latter property in the form of a Stieltjes integral:

$$\int_0^\infty Z(t)\mathrm{d}L(t) = 0.$$

One way to think about this is that $\mathrm{d}L$ is the *minimal amount of push* required to keep Brownian motion $W$ to the right of zero. When $W$ "wants" to wander into the negative half-line, we "help" it stay on the positive half-line, but the amount of this "help" is as small as possible. The process $L$ is called the *local time of the Brownian motion $W$ at zero*; its equivalent representation (which is usually taken as the definition) is

$$L(t) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \,\mathrm{mes}\left\{s \in [0, t] \mid -\varepsilon \le W(s) \le \varepsilon\right\}.$$

(Sometimes they refer to the process $(1/2)L$, rather than $L$, as the local time.) Respresentation (2.1), which is known as *Tanaka formula*, can be found in standard stochastic calculus textbooks, such as [73, Proposition 3.6.8], or [97, Chapter 6, Theorem 1.2]. Consider now a deterministic analogue of the Tanaka formula, which is called the *Skorohod problem in the positive half-line*.

**Definition 1.** Take a continuous function $\mathcal{X} : \mathbb{R}_+ \to \mathbb{R}$ with $\mathcal{X}(0) \ge 0$. A *solution to the Skorohod problem in* $\mathbb{R}_+$ *with driving function* $\mathcal{X}$ is a continuous function $\mathcal{Z} : \mathbb{R}_+ \to \mathbb{R}_+$ such that there exists another continuous function $\mathcal{L} : \mathbb{R}_+ \to \mathbb{R}$ with the following properties:

(i) $\mathcal{X}(t) = \mathcal{Z}(t) + \mathcal{L}(t), \quad t \ge 0$;

(ii) $\mathcal{L}$ is a nondecreasing function with $\mathcal{L}(0) = 0$, which can increase only when $\int_0^\infty \mathcal{Z}(t)\mathrm{d}\mathcal{L}(t) = 0$.

The function $\mathcal{L}$ is called the *boundary term*.

The following theorem was proved in the articles [107], [108], which pioneered the study of reflected Brownian motion.

**Proposition 2.1.1.** *For every continuous function* $\mathcal{X} : \mathbb{R}_+ \to \mathbb{R}$ *with* $\mathcal{X}(0) \ge 0$, *there exists a unique solution to the Skorohod problem in* $\mathbb{R}_+$ *with driving function* $\mathcal{X}$, *which is given by the formula*

$$\mathcal{Z}(t) = \mathcal{X}(t) - \mathcal{L}(t), \quad \mathcal{L}(t) := \max_{[0,t]} \left[-\mathcal{X}(s)\right]_+ .$$

Next, we move to the multidimensional version of the Skorohod problem.

## 2.2 The Skorohod Problem in the Orthant

The Skorohod problem in the orthant has one feature which distinguishes it from its one-dimensional version: *direction of reflection matters.* That is, in the one-dimensional case we had only one possible direction of reflection: rightward, back to the positive half-line $\mathbb{R}_+$. Now, consider the multidimensional positive orthant $\mathbb{R}^d_+$ instead of $\mathbb{R}_+$. As the Brownian motion (or any other driving function) hits a face of the boundary, it can be reflected *normally* as well as *obliquely.* In the following definition, we make this observation rigorous.

**Definition 2.** Take a continuous function $\mathcal{X} : \mathbb{R}_+ \to \mathbb{R}^d$ with $\mathcal{X}(0) \in S$. A *solution to the Skorohod problem in the positive orthant* $S$ *with reflection matrix* $R$ *and driving function* $\mathcal{X}$

is a continuous function $\mathcal{Z} : \mathbb{R}_+ \to S$ such that there exists another continuous function $\mathcal{L} : \mathbb{R}_+ \to \mathbb{R}^d$ with the following properties:

(i) for every $t \geq 0$, we have: $\mathcal{Z}(t) = \mathcal{X}(t) + R\mathcal{L}(t)$;

(ii) for every $i = 1, \ldots, d$, the function $\mathcal{L}_i$ is nondecreasing, satisfies $\mathcal{L}_i(0) = 0$ and can increase only when $\mathcal{Z}_i(t) = 0$, that is, when $\mathcal{Z}(t) \in S_i$. We can write the last property formally as $\int_0^\infty \mathcal{Z}_i(t)\mathrm{d}\mathcal{L}_i(t) = 0$.

The function $\mathcal{L}$ is called the *vector of boundary terms*, and its component $\mathcal{L}_i$ is called the *boundary term*, corresponding to the face $S_i$, for $i = 1, \ldots, d$.

*Remark* 1. This definition can also be stated for a finite time horizon, that is, for functions $\mathcal{X}, \mathcal{L}, \mathcal{Z}$ defined on $[0, T]$ instead of $\mathbb{R}_+$.

For which matrices $R$ do we have existence and uniqueness of a solution to the Skorohod problem? We need to introduce some definitions.

**Definition 3.** Take a $d \times d$-matrix $R = (r_{ij})_{1 \leq i,j \leq d}$. It is called a *reflection matrix* if $r_{ii} = 1$ for $i = 1, \ldots, d$. It is called *nonnegative* if all its elements are nonnegative, that is, if $R \geq 0$; it is called *strictly nonnegative* if it is nonnegative and $r_{ii} > 0$ for $i = 1, \ldots, d$. It is called an *$\mathcal{S}$-matrix* if there exists a vector $u \in \mathbb{R}^d,\ u > 0$ such that $Ru > 0$. Any submatrix of $R$ of the form $[R]_I$, where $I \subseteq \{1, \ldots, d\}$ is a nonempty subset, is called a *principal submatrix* (this includes the matrix $R$ itself). The matrix $R$ is called *completely-$\mathcal{S}$* if each of its principal submatrices is an $\mathcal{S}$-matrix. It is called a $\mathcal{Z}$-matrix if $r_{ij} \leq 0$ for $i \neq j$. It is called *strictly inverse-nonnegative* if it is invertible and its inverse $R^{-1}$ is a strictly nonnegative matrix. It is called a *nonsingular $\mathcal{M}$-matrix* if it is both completely-$\mathcal{S}$ and a $\mathcal{Z}$-matrix.

The following lemma is a useful characterization of reflection nonsingular $\mathcal{M}$-matrices.

**Lemma 2.2.1.** *Suppose $R$ is a $d \times d$ reflection matrix. Then the following statements are equivalent:*

*(i) $R$ is a nonsingular $\mathcal{M}$-matrix;*

*(ii) $R$ is a strictly inverse-nonnegative $\mathcal{Z}$-matrix;*

*(iii) $R = I_d - Q$, where $Q$ is a nonnegative matrix with spectral radius less than* 1.

*Proof.* (i) $\Rightarrow$ (iii). Use [55, Theorem 2.5.3]. Since $R$ is completely-$\mathcal{S}$, it satisfies condition 12 from this theorem. Therefore, it satisfies condition 2 from this theorem. We get the following representation: $R = \gamma I_d - Q$, where $\gamma := \max_{1\le i\le d} r_{ii} = 1$, and a $d\times d$-matrix $Q$ is nonnegative with spectral radius less than one. (See the beginning of [55, Section 2.5.4].)

(iii) $\Rightarrow$ (ii). By [85, Section 7.10], we can represent $R^{-1}$ as Neumann series:

$$R^{-1} = I_d + Q + Q^2 + \dots$$

Since $Q$ is nonnegative, $R^{-1}$ is also nonnegative, and the diagonal elements of $R^{-1}$ are strictly positive (and even greater than or equal to 1).

(ii) $\Rightarrow$ (i). Apply [55, Theorem 2.5.3] again: condition 17 implies condition 12. Therefore, there exists $x \in \mathbb{R}^d$, $x > 0$ such that $Rx > 0$, so $R$ is an $\mathcal{S}$-matrix. Take a principal submatrix $\tilde{R}$ of $R$ and show that it is also an $\mathcal{S}$-matrix. Let $\tilde{R} := [R]_I$, where $I \subsetneq \{1, \dots, d\}$ is a nonempty set. Let $\tilde{x} := [x]_I$. Then $r_{ij} \le 0$ for $i \in I$ and $j \in I^c := \{1, \dots, d\} \setminus I$, and

$$\left(\tilde{R}\tilde{x}\right)_i = \sum_{j\in I} r_{ij}x_j \ge \sum_{i=1}^{d} r_{ij}x_j = (Rx)_i > 0, \quad i \in I.$$

Therefore, $\tilde{x} > 0$ and $\tilde{R}\tilde{x} > 0$. So every principal submatrix of $R$ is an $\mathcal{S}$-matrix, which proves that the matrix $R$ is completely-$\mathcal{S}$. $\square$

Now, we can formulate the main existence and uniqueness result for the Skorohod problem, proved in [51, Theorem 1], see also [125, Theorem 2.1].

**Proposition 2.2.2.** *Suppose $R$ is a $d\times d$ reflection nonsingular $\mathcal{M}$-matrix. Then for every continuous driving function $\mathcal{X} : \mathbb{R}_+ \to \mathbb{R}^d$ with $\mathcal{X}(0) \in S$, the Skorohod problem in the orthant $S$ with reflection matrix $R$ has a unique solution.*

### *2.3 SRBM in the Orthant: Definition, Existence and Uniqueness Results*

Let us rigorously define an SRBM in the orthant. Take the parameters $R, A, \mu$, described above. Assume the usual setting: a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbf{P})$ with the filtration satisfying the usual conditions.

**Definition 4.** Suppose $B = (B(t), t \geq 0)$ is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion in $\mathbb{R}^d$ with drift vector $\mu$ and covariance matrix $A$. A solution $Z = (Z(t), t \geq 0)$ to the Skorohod problem in $S$ with reflection matrix $R$ and driving function $B$ is called a *semimartingale reflected Brownian motion*, or SRBM, *in the positive orthant* $S$ with *reflection matrix* $R$, *drift vector* $\mu$ and *covariance matrix* $A$. It is denoted by $\mathrm{SRBM}^d(R, \mu, A)$. The process $B$ is called the *driving Brownian motion*. We say that $Z$ *starts from* $x \in S$ if $Z(0) = x$ a.s.

We are ready to state an existence and uniqueness result, shown in [51]. (It was shown in this article for a slightly more resitricted case, but it can be readily generalized for the given conditions.) This is not the most general result (which we shall discuss a bit later), but it is sufficient for our purposes.

**Proposition 2.3.1.** *Suppose $R$ is a $d \times d$ reflection nonsingular $\mathcal{M}$-matrix. Take a vector $\mu \in \mathbb{R}^d$ and a $d \times d$ positive definite symmetric matrix $A$. For every $x \in S$, there exists in the strong sense an* $\mathrm{SRBM}^d(R, \mu, A)$ *starting from $x$, and this process is pathwise unique. These processes, starting from different $x \in S$, form a Feller continuous strong Markov family.*

The part about existence and uniqueness follows immediately from Proposition 2.2.2. Now, for the sake of completeness, let us state the most general result, for which the reader might want to see [96], [115], [18, Theorem 3.1, Theorem 3.2], and [125, Theorem 2.3]. However, we shall not need this result in our thesis.

**Proposition 2.3.2.** *Take a drift vector $\mu \in \mathbb{R}^d$ and a positive definite symmetric $d \times d$-matrix $A$. Fix a starting point $x \in S$. Then the* $\mathrm{SRBM}^d(R, \mu, A)$*, starting from $x$, exists in a weak sense if and only if $R$ is completely-$\mathcal{S}$. In this case, this process is pathwise unique. These processes, starting from different $x \in S$, form a Feller continuous strong Markov family.*

## 2.4 Stationary Distributions and Convergence

Consider the process $Z = (Z(t), t \geq 0)$, which is an $\mathrm{SRBM}^d(R, \mu, A)$ in the orthant $S = \mathbb{R}^d_+$. If it starts from $x \in S$, then we denote the corresponding probability measure by $\mathbf{P}_x$, and the corresponding expectation by $\mathbf{E}_x$. This is a standard notation in probability. Note that we can also start $Z$ from a distribution $Z(0) \backsim \pi$ on $S$, rather than a fixed point $x \in S$. Such process exists and is unique in a strong sense (a trivial corollary of Proposition 2.3.1). We denote the corresponding probability measure and expectation by $\mathbf{P}_\pi$ and $\mathbf{E}_\pi$.

**Definition 5.** We say that the distribution $\pi$ on $S$ is a *stationary distribution* for the process $Z$ if

$$Z(0) \backsim \pi \ \Rightarrow \ Z(t) \backsim \pi \ \text{ for every } \ t \ge 0.$$

In other words, if we start the SRBM from the initial distribution $Z(0) \backsim \pi$, then at every moment this process has the same distribution $Z(t) \backsim \pi$.

**Definition 6.** Assume that $Z$ has a stationary distribution $\pi$. It is *ergodic* if this distribution is unique and for every $x \in S$ we have:

$$\|P^t(x, \cdot) - \pi(\cdot)\|_{\mathrm{TV}} \to 0, \quad t \to \infty.$$

We say that $Z$ is *exponentially ergodic* if there exists $\varkappa > 0$ such that for every $x \in S$ we have:

$$\|P^t(x, \cdot) - \pi(\cdot)\|_{\mathrm{TV}} \le C(x)e^{-\varkappa t},$$

where $C(x) > 0$.

In this section, we shall enunciate a few known facts about stationary distributions and convergence. We refer the reader to a comprehensive survey [125].

The following result was proved in [52, Theorem 7.1, Theorem 8.1(i)], see also the aforementioned survey [125, Lemma 3.1(i), (ii)].

**Proposition 2.4.1.** *If the process $Z$ has a stationary distribution, then this stationary distribution is unique. Moreover, for each $i = 1, \ldots, d$, there exists a finite Borel measure $\nu_i$ on the face $S_i$ such that for every bounded Borel measurable $f : S_i \to \mathbb{R}$ and for every $t \ge 0$ we have:*

$$\mathbf{E}_\pi \left[\int_0^t f(Z(s))\mathrm{d}Y_i(s)\right] = t\int_{S_i} f(x)\mathrm{d}\nu_i(x).$$

These $\nu_1, \ldots, \nu_d$ are called *boundary measures* corresponding to the stationary distribution $\pi$. Now, let us state an equivalent characterization of existence of a stationary distribution, which is useful in the theory of Markov processes. For a subset $A \subseteq S$, let $\tau_A := \inf\{t \geq 0 \mid Z(t) \in A\}$ be the hitting time of $A$.

**Definition 7.** We say that the process $Z$ is *positive recurrent* if for every $x \in S$ and every closed $A \subseteq S$ with positive Lebesgue measure we have:

$$\mathbf{E}_x \tau_A < \infty.$$

Informally, the process is positive recurrent if it visits every "sufficiently large" set, and the visit occurs "not too late". The following fact follows from the general theory of Markov processes and is proved in [18].

**Proposition 2.4.2.** *The process $Z$ has a stationary distribution if and only if it is positive recurrent.*

In this subsection, we note that if the process $Z$ is positive recurrent, then it converges to its stationary distribution exponentially fast. Let $P^t(x, A) \equiv \mathbf{P}_x(Z(t) \in A)$ be the transition function of $Z$. Then $P^t(x, \cdot)$ is a probability distribution on $S$.

**Proposition 2.4.3.** *If $Z$ is positive recurrent, then it is ergodic.*

The following proposition was proved in [18, Theorem 3.4].

**Proposition 2.4.4.** *Suppose that $Z$ is positive recurrent. Then $R$ is invertible and $R^{-1}\mu < 0$.*

**Definition 8.** We say that $Z$ satisfies the *fluid path condition* if for every $x \in S$, the solution $z(t)$ to the Skorohod problem with driving function $x + \mu t$ and reflection matrix $R$ has the property $\lim_{t\to\infty} z(t) = 0$.

**Proposition 2.4.5** ([22], [9])**.** *Suppose that $Z$ satisfies the fluid path condition. Then the process $Z$ is positive recurrent and exponentially ergodic.*

**Proposition 2.4.6** ([12])**.** *If $R$ is a nonsingular $\mathcal{M}$-matrix and $R^{-1}\mu < 0$, then $Z$ satisfies the fluid path condition.*

Let us summarize results for a special case when $R$ is a nonsingular $\mathcal{M}$-matrix. This is the case which it used for competing Brownian particles. The next corollary is an immediate consequence of Propositions 2.4.4, 2.4.5, and 2.4.6.

**Corollary 2.4.7.** *Suppose $R$ is a nonsingular $\mathcal{M}$-matrix. Then $Z$ is positive recurrent (or, equivalently, has a stationary distribution) if and only if $R^{-1}\mu < 0$. In this case, this stationary distribution is unique and the process $Z$ is exponentially ergodic.*

The next proposition was proved for $d = 2$ in [54] and for $d = 3$ in [8].

**Proposition 2.4.8.** *For dimensions $d = 2$ and $d = 3$, the fluid path condition is not only sufficient but necessary for positive recurrence. Therefore, in these dimensions the fluid path condition implies that $R$ is invertible and $R^{-1}\mu < 0$.*

*Remark* 2. In dimension $d = 2$, the fluid path condition is equivalent to $R^{-1}\mu < 0$. In dimension $d = 3$, this is no longer true; see [8].

For a function $f : S \to \mathbb{R}$, define

$$D_i f(x) \equiv r_i \cdot \nabla f(x), \quad i = 1, \dots, d; \quad \mathcal{A}f(x) := \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}\frac{\partial^2 f(x)}{\partial x_i \partial x_j}.$$

**Definition 9.** Take a probability distribution $\pi$ on $S$ and finite Borel measures $\nu_1, \dots, \nu_d$ on $S_1, \dots, S_d$. We say that this collection $(\pi, \nu_1, \dots, \nu_d)$ *satisfies the Basic Adjoint Relationship* if for every $f \in C_b^2(S)$ we have:

$$\int_S \mathcal{A}f(x)\mathrm{d}\pi(x) + \sum_{i=1}^{d}\int_{S_i} D_i f(x)\mathrm{d}\nu_i(x) = 0.$$

**Theorem 2.4.9** ([16])**.** *If $\pi$ is a stationary distribution, then it satisfies the Basic Adjoint Relationship together with the corresponding boundary measures. Conversely, if $(\pi, \nu_1, \dots, \nu_d)$ satisfy the Basic Adjoint Relationship, then $\pi$ is the stationary distribution, and $\nu_1, \dots, \nu_d$ are corresponding boundary measures.*

**Definition 10.** We say that the process $Z$ satisfies the *skew-symmetry condition* if

$$r_{ij}a_{jj} + r_{ji}a_{ii} = 2r_{ij}, \quad 1 \le i < j \le d.$$

We can also write it as follows:

$$RD + DR' = 2A,$$

where $D = \mathrm{diag}(A)$ is the diagonal $d \times d$-matrix with the same diagonal entries as $A$.

**Definition 11.** We say that a distribution $\pi$ on $S$ has *product form* if for some distributions $\pi_1, \dots, \pi_d$ on $\mathbb{R}_+$ we have:

$$\pi(\mathrm{d}x) = \bigotimes_{i=1}^{d} \pi_i(\mathrm{d}x_i), \quad x = (x_1, \dots, x_d)'.$$

**Proposition 2.4.10** ([52])**.** *Assume that $R$ is a nonsingular $\mathcal{M}$-matrix and $b := R^{-1}\mu < 0$. The stationary distribution $\pi$ of $Z$ has product form if and only if it satisfies the skew-symmetry condition. In this case,*

$$\pi = \bigotimes_{i=1}^{d} \mathcal{E}\left(2a_{ii}^{-1}b_i\right).$$

Let us also mention a comparison result from [76], see also [74], [75], [78], [93], [77]. This is part of Theorem 4.2.1 from Chapter 4: the part concerning solutions of the Skorohod problem. We reprove it in this thesis, and also prove the other part of this theorem concerning boundary terms.

### *2.5 Motivation and Literature Review*

As mentioned in Section 2.1, the study of the reflected Brownian motion started in the papers [107], [108], see also the book [43], and an article [82]. Multidimensional (normally and obliquely) reflected Brownian motion in general regions was also studied in many other articles, including [114] (convex regions), [119], [120], [81], [111], as well as [51], [53], [52], [125] (positive multidimensional orthant).

Studying an SRBM in the orthant is motivated by queueing theory. An SRBM in the orthant is the *heavy traffic limit* for series of queues, when the traffic intensity at each queue tends to one, see [94], [95], [47], [46]; see also related works [48], [62], [80]. We can also define an SRBM in general convex polyhedral domains in $\mathbb{R}^d$, see [17]. An SRBM in the orthant and in convex polyhedra has been extensively studied, see the survey [125]. A special case of a convex polyhedron is a two-dimensional wedge, see [121], [122], [116], [123].

An SRBM in the orthant was introduced and defined in [51] and [52]. Stationary distributions were found in [50], [53], [124], [16] (the two latter papers also study the case of convex polyhedra). General existence and uniqueness result was proved in [96] (necessaity) and [115] (sufficiency). The fluid path condition for positive recurrence was established in [22], see also [12] for some simpler sufficient conditions (which are stronger than the fluid path condition). The fluid path condition is not only sufficient but necessary for positive recurrence in dimensions $d = 2$, see [54], and $d = 3$, see [8] (and also related papers [15], [23]). Under the fluid path condition, an SRBM is not only positive recurrent but converges exponentially fast to the stationary distribution, see [9]. Some properties of the stationary distribution in two dimensions were studied in [49].

An invariance principle for an SRBM in the orthant was formulated in [126], and for more general cases in [127]. Numerical methods for finding the stationary distribution are studied in [14]. Comparison techniques similar to ones discussed in Chapter 4 (which corresponds to the author's paper [100]) are studied in [74], [75], [78], [93], [76], [77].

The Ph.D. thesis [18, Chapter 3] by Jim Dai is a useful collection of facts about an SRBM in the orthant (some of them with proofs).

The Ph.D. thesis [88, Chapter 5] by Janosch Ortmann deals with a *generalized reflected Brownian motion* in a polyhedral domain, which is a solution to an SDE. See also the paper [86].

# Chapter 3
# COMPETING BROWNIAN PARTICLES

This chapter is organized as follows. First, we define classical systems of competing Brownian particles, where particles are presumed to have “the same mass”; that is, when two particles collide, the local time of collision is split evenly between them. Then, we modify this model to allow for different prpportions of split; these systems have *asymmetric collisions*. We then introduce a deterministic analogue of competing Brownian particles: it is called simply a *system of competing particles*. It bears the same relation to competing Brownian particles as the Skorohod problem to an SRBM. We show that the gap process for systems of competing particles is actually a solution to the Skorohod problem in the orthant. Next, we state some (already known) properties and results for competing Brownian particles. Then we move to infinite classical systems of competing Brownian particles. We state the definition and outline some already known results.

Then we discuss the McKean-Vlasov equation, which is a continuous analogue of a system of competing Brownian particles. Next, we give a brief introduction to *Stochastic Portfolio Theory*, a newly developed area of Financial Mathematics, and mention its connections with the theory of competing Brownian particles. Finally, we conduct a literature review and mention some connections to other areas of probability.

## 3.1 Classical Systems of Competing Brownian Particles

In this subsection, we use definitions from [3]. Assume the usual setting: a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbf{P})$ with the filtration satisfying the usual conditions. Let $N \ge 2$ (the number of particles). Fix parameters

$$g_1, \dots, g_N \in \mathbb{R}; \quad \sigma_1, \dots, \sigma_N > 0.$$

We wish to define a system of $N$ Brownian particles in which the $k$th smallest particle moves a Brownian motion with drift $g_k$ and diffusion $\sigma_k^2$. We resolve ties in the lexicographic order, as described in the Introduction.

**Definition 12.** Take i.i.d. standard $(\mathcal{F}_t)_{t\ge 0}$-Brownian motions $W_1, \ldots, W_N$. For a continuous $\mathbb{R}^N$-valued process $X = (X(t),\ t \ge 0)$, $X(t) = (X_1(t), \ldots, X_N(t))'$, let us define $\mathbf{p}_t,\ t \ge 0$, the *ranking permutation* for the vector $X(t)$: this is a permutation on $\{1, \ldots, N\}$ such that:

(i) $X_{\mathbf{p}_t(i)}(t) \le X_{\mathbf{p}_t(j)}(t)$ for $1 \le i < j \le N$;

(ii) if $1 \le i < j \le N$ and $X_{\mathbf{p}_t(i)}(t) = X_{\mathbf{p}_t(j)}(t)$, then $\mathbf{p}_t(i) < \mathbf{p}_t(j)$. (This permutation always exists and is unique.)

Suppose the process $X$ satisfies the following SDE:

$$dX_i(t) = \sum_{k=1}^{N} 1(\mathbf{p}_t(k) = i)\left[g_k\,\mathrm{d}t + \sigma_k\,\mathrm{d}W_i(t)\right], \quad i = 1, \ldots, N. \tag{3.1}$$

Then this process $X$ is called a *classical system of $N$ competing Brownian particles* with *drift coefficients* $g_1, \ldots, g_N$ and *diffusion coefficients* $\sigma_1^2, \ldots, \sigma_N^2$. For $i = 1, \ldots, N$, the component $X_i = (X_i(t), t \ge 0)$ is called the *$i$th named particle*. For $k = 1, \ldots, N$, the process

$$Y_k = (Y_k(t),\ t \ge 0),\quad Y_k(t) := X_{\mathbf{p}_t(k)}(t) \equiv X_{(k)}(t),$$

is called the *$k$th ranked particle*. They satisfy $Y_1(t) \le Y_2(t) \le \ldots \le Y_N(t)$, $t \ge 0$. If $\mathbf{p}_t(k) = i$, then we say that the particle $X_i(t) = Y_k(t)$ at time $t$ has *name* $i$ and *rank* $k$.

The coefficients of the SDE (3.1) are piecewise constant functions of $X_1(t), \ldots, X_N(t)$, so weak existence and uniqueness in law for such systems follow from [6].

A particular case

$$g_1 = 1,\ g_2 = \ldots = g_N = 0,\ \sigma_1 = \ldots = \sigma_N = 1$$

is called *the Atlas model*.

**Definition 13.** A *triple collision at time $t$* occurs if there exists a rank $k = 2, \ldots, N-1$ such that $Y_{k-1}(t) = Y_k(t) = Y_{k+1}(t)$.

The following result was proved in [59].

**Theorem 3.1.1.** *If $\tau$ is the first moment of a triple collision, then the classical system of competing Brownian particles has strong existence and pathwise uniqueness up to the moment $\tau$. In particular, if there are a.s. no triple collisions at any time $t \geq 0$, then strong existence and pathwise uniqueness hold on the infinite time horizon.*

The question whether strong solution exists after the first triple collision is an open problem.

Now, let us find an equation for the *ranked particles* $Y_k$. Define the processes $B_1 = (B_1(t), t \geq 0), \ldots, B_N = (B_N(t), t \geq 0)$ as follows:

$$B_k(t) = \sum_{i=1}^{N} \int_0^t 1(\mathbf{p}_s(k) = i) \mathrm{d}W_i(s).$$

One can calculate that $\langle B_i, B_j \rangle_t = \delta_{ij} t$; therefore, these are i.i.d. standard Brownian motions. For $k = 2, \ldots, N$, let the process $L_{(k-1,k)} = (L_{(k-1,k)}(t),\ t \geq 0)$ be the semimartingale local time at zero of the nonnegative semimartingale $Y_k - Y_{k-1}$. For notational convenience, we let $L_{(0,1)}(t) \equiv 0$ and $L_{(N,N+1)}(t) \equiv 0$. Then the ranked particles $Y_1, \ldots, Y_N$ satisfy the following equation:

$$Y_k(t) = Y_k(0) + g_k t + \sigma_k B_k(t) + \frac{1}{2} L_{(k-1,k)}(t) - \frac{1}{2} L_{(k,k+1)}(t), \quad k = 1, \ldots, N. \tag{3.2}$$

The equation (3.2) was deduced in [2, Lemma 1] and [4, Theorem 2.5]; see also [3, Section 3] and [57, Chapter 3].

The process $L_{(k-1,k)}$ is called the *local time of collision between the particles $Y_{k-1}$ and $Y_k$*. One can regard the local time $L_{(k-1,k)}(t)$ to be the total amount of push between the $(k-1)$st and the $k$th ranked particles $Y_{k-1}$ and $Y_k$ accumulated by time $t$. This amount of push is necessary and sufficient to keep the particle $Y_k$ to the right of the particle $Y_{k-1}$, so that $Y_{k-1}(t) \le Y_k(t)$. Indeed, "left to themselves", the particles $Y_{k-1}$ and $Y_k$ "want" to move as Brownian motions, which will eventually clearly violate the condition $Y_{k-1}(t) \le Y_k(t)$.

When these two particles collide, the amount of push is split evenly between them: the amount $(1/2)L_{(k-1,k)}(t)$ goes to the right-sided particle $Y_k$ and pushes it to the right; the equal amount $(1/2)L_{(k-1,k)}(t)$ (with the minus sign) goes to the left-sided particle $Y_{k-1}$ and pushes it to the left. One possible physical interpretation of this phenomenon: the ranked particles have the same mass; so, when they collide, they get the same amount of push.

The local time process $L_{(k-1,k)}$ has the following properties: $L_{(k-1,k)}(0) = 0$, $L_{(k-1,k)}$ is nondecreasing, and it can increase only when $Y_{k-1}(t) = Y_k(t)$, that is, when particles with ranks $k-1$ and $k$ collide. We can formally write the last property as

$$\int_0^{\infty} 1(Y_k(t) \ne Y_{k-1}(t))\mathrm{d}L_{(k-1,k)}(t) = 0. \tag{3.3}$$

### 3.2 Systems of Competing Brownian Particles with Asymmetric Collisions

If we change coefficients 1/2 in (3.2) to some other values, we get the model from the paper [71]. The local times in this new model are split unevenly between the two colliding particles, as if they had different mass. Let us now formally define this model. First, let us describe its parameters. Let $N \ge 2$ be the quantity of particles. Fix real numbers $g_1, \ldots, g_N$ and positive real numbers $\sigma_1, \ldots, \sigma_N$, as before. In addition, fix real numbers $q_1^+$, $q_1^-, \ldots, q_N^+$, $q_N^-$, satisfying the following conditions:

$$q_{k+1}^+ + q_k^- = 1, \quad k = 1, \ldots, N-1; \quad 0 < q_k^{\pm} < 1, \quad k = 1, \ldots, N.$$

**Definition 14.** Take i.i.d. standard $(\mathcal{F}_t)_{t\ge 0}$-Brownian motions $B_1, \ldots, B_N$. Consider a continuous adapted $\mathbb{R}^N$-valued process

$$Y = (Y(t),\ t \ge 0), \quad Y(t) = (Y_1(t), \ldots, Y_N(t))',$$

and $N-1$ continuous adapted real-valued processes

$$L_{(k-1,k)} = (L_{(k-1,k)}(t),\ t \ge 0), \quad k = 2, \ldots, N,$$

with the following properties:

(i) $Y_1(t) \le \ldots \le Y_N(t), \quad t \ge 0,$

(ii) the process $Y$ satisfies the following system of equations:

$$Y_k(t) = Y_k(0) + g_k t + \sigma_k B_k(t) + q_k^+ L_{(k-1,k)}(t) - q_k^- L_{(k,k+1)}(t), \quad k = 1, \ldots, N. \tag{3.4}$$

We let $L_{(0,1)}(t) \equiv 0$ and $L_{(N,N+1)}(t) \equiv 0$ for notational convenience.

(iii) for each $k = 2, \ldots, N$, the process $L_{(k-1,k)} = (L_{(k-1,k)}(t),\ t \ge 0)$ has the properties mentioned above: $L_{(k-1,k)}(0) = 0$, $L_{(k-1,k)}$ is nondecreasing and satisfies (3.3).

Then the process $Y$ is called *a system of $N$ competing Brownian particles with asymmetric collisions*, with *drift coefficients* $g_1, \ldots, g_N$, *diffusion coefficients* $\sigma_1^2, \ldots, \sigma_N^2$, and *parameters of collision* $q_1^{\pm}, \ldots, q_N^{\pm}$. For each $k = 1, \ldots, N$, the process $Y_k = (Y_k(t), t \ge 0)$ is called the *$k$th ranked particle*. For $k = 2, \ldots, N$, the process $L_{(k-1,k)}$ is called the *local time of collision between the particles $Y_{k-1}$ and $Y_k$*.

The state space of the process $Y$ is

$$\mathcal{W}^N := \{y = (y_1, \ldots, y_N)' \in \mathbb{R}^N \mid y_1 \le y_2 \le \ldots \le y_N\}.$$

Strong existence and pathwise uniqueness for $Y$ and $L$ are proved in [71, Section 2.1]; they also follow from Lemma 3.4.3 below.

*Remark* 3. Triple and simultaneous collisions for these systems are defined similarly to Definitions 22 and 23.

In the case of asymmetric collisions, we can also define a corresponding named system of competing Brownian particles.

**Definition 15.** Consider a continuous adapted process

$$X = (X(t), t \ge 0),\ X(t) = (X_1(t), \ldots, X_N(t))'.$$

Suppose $\mathbf{p}_t$ is the ranking permutation of $X(t)$ for $t \ge 0$, as before, and

$$Y_k(t) \equiv X_{\mathbf{p}_k(t)}(t), \quad k = 1, \ldots, N,\ t \ge 0,$$

Let $L_{(k-1,k)} = (L_{(k-1,k)}(t), t \ge 0)$ be the semimartingale local time at zero of $Y_k - Y_{k-1}$, for $k = 2, \ldots, N$; and $L_{(0,1)}(t) \equiv L_{(N,N+1)}(t) \equiv 0$ for notational convenience, as before.

Then this system $X = (X_1, \ldots, X_N)'$ is governed by the following SDE: for $i = 1, \ldots, N$ and $t \ge 0$,

$$\begin{aligned}\mathrm{d}X_i(t) = &\sum_{k=1}^{N} 1(\mathbf{p}_t(k) = i)\,(g_k\mathrm{d}t + \sigma_k \mathrm{d}W_i(t)) \\ &+ \sum_{k=1}^{N} 1(\mathbf{p}_t(k) = i)\left(q_k^- - (1/2)\right)\mathrm{d}L_{(k,k+1)}(t) \\ &+ \sum_{k=1}^{N} 1(\mathbf{p}_t(k) = i)\left(q_k^+ - (1/2)\right)\mathrm{d}L_{(k-1,k)}(t).\end{aligned}$$

It is called a *system of named competing Brownian particles* with *drift coefficients* $(g_n)_{1\le n\le N}$, *diffusion coefficients* $(\sigma_n^2)_{1\le n\le N}$, and *parameters of collision* $(q_n^{\pm})_{1\le n\le N}$.

The ranked particles $(Y_1, \ldots, Y_N)$ from Definition 15 form a system of ranked competing Brownian particles in the sense of Definition 14. However, unlike the system $Y$ from Definition 14, which exists and is unique in a strong sense up to the infinite time horizon, the system $X$ from Definition 15 is known to have strong solutions only up to the first moment of a triple collision, see [71]. This provides a motivation to find a condition which guarantees absense of triple collisions. Here, we prove a necessary and sufficient condition for a.s. lack of triple collisions.

### *3.3 General Systems of (non-Brownian) Competing Particles*

As mentioned in the Introduction, in this chapter we consider not just systems of competing Brownian particles, but more general systems, with arbitrary continuous functions instead of Brownian motions. These general systems are called *systems of competing particles*; they might be random or deterministic. Let us now define them.

**Definition 16.** Fix a continuous function $X = (X_1, \ldots, X_N)' : \mathbb{R}_+ \to \mathbb{R}^N$ such that $X(0) \in \mathcal{W}_N$. Take *parameters of collision*: real numbers $q_1^+, q_1^-, \ldots, q_N^+, q_N^-$ which satisfy (3.5). Consider a continuous function $Y = (Y_1, \ldots, Y_N)' : \mathbb{R}_+ \to \mathcal{W}_N$, and other $N-1$ continuous functions $L_{(1,2)}, \ldots, L_{(N-1,N)} : \mathbb{R}_+ \to \mathbb{R}$ such that:

(i) $Y_k(t) = X_k(t) + q_k^+ L_{(k-1,k)}(t) - q_k^- L_{(k,k+1)}(t)$ for $k = 1, \ldots, N$ and $t \ge 0$ (we let $L_{(0,1)}(t) \equiv 0$ and $L_{(N,N+1)}(t) \equiv 0$ for notational convenience);

(ii) $L_{(k,k+1)}(0) = 0$ for $k = 1, \ldots, N-1$;

(iii) $L_{(k,k+1)}$ is nondecreasing for each $k = 1, \ldots, N-1$;

(iv) $L_{(k,k+1)}$ can increase only when $Y_k(t) = Y_{k+1}(t)$; we can write this formally as the following Stieltjes integral:

$$\int_0^\infty (Y_{k+1}(t) - Y_k(t))\, \mathrm{d}L_{(k,k+1)}(t) = 0, \quad k = 1, \ldots, N-1.$$

Then the function $Y$ is called a *system of $N$ competing particles* with *driving function $X$* and *parameters of collisions* $(q_k^\pm)_{1\le k\le N}$. The $k$th component $Y_k$ of the function $Y$ is called the *$k$th ranked particle*. The function $L_{(k,k+1)}$ is called the *collision term* between the $k$th and the $k+$ 1st ranked particles $Y_k$ and $Y_{k+1}$. The vector-valued function $L = (L_{(1,2)}, L_{(2,3)}, \ldots, L_{(N-1,N)})'$ is called the *vector of collision terms*. We say that this system *starts with $y$*, if $Y(0) = y$. The *gap process* is defined as was already shown in the Introduction: this is an $\mathbb{R}_+^{N-1}$-valued process

$$Z = (Z(t),\ t \ge 0),\ Z(t) = (Z_1(t), \ldots, Z_{N-1}(t))',$$

$$Z_k(t) = Y_{k+1}(t) - Y_k(t), \quad k = 1, \ldots, N-1, \quad t \ge 0.$$

Now, for the sake of completeness, we essentially rephrase Definition 14, tying systems of competing *Brownian* particles to general systems of competing particles.

**Definition 17.** Assume the standard probabilistic setting: a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbf{P})$, with the filtration satisfying the usual conditions. Take i.i.d. standard $(\mathcal{F}_t)_{t\ge0}$-Brownian motions $B_1, \ldots, B_N$ and an $\mathcal{F}_0$-measurable random vector $y \in \mathcal{W}_N$. Fix parameters of collision $q_1^+, q_1^-, \ldots, q_N^+, q_N^-$ which satisfy

$$q_{n+1}^+ + q_n^- = 1, \quad 0 < q_n^\pm < 1, \quad n = 1, 2, \ldots \tag{3.5}$$

Also, fix real numbers $g_1, \ldots, g_N$ and positive real numbers $\sigma_1, \ldots, \sigma_N$. Consider a system $Y$ of $N$ competing particles with the driving function

$$\mathcal{X} = (\mathcal{X}_1, \ldots, \mathcal{X}_N)', \quad \mathcal{X}_k(t) = y_k + g_k t + \sigma_k B_k(t), \quad k = 1, \ldots, N, \quad t \geq 0,$$

and parameters of collision $(q_k^{\pm})_{1 \leq k \leq N}$. Then $Y$ is called a (*ranked*) *system of competing Brownian particles* with *drift coefficients* $g_1, \ldots, g_N$, *diffusion coefficients* $\sigma_1, \ldots, \sigma_N$, and *parameters of collision* $(q_k^{\pm})_{1 \leq k \leq N}$. The standard Brownian motions $B_1, \ldots, B_N$ are called the *driving Brownian motions*. For each $k = 1, \ldots, N-1$, the collision term $L_{(k,k+1)}$ is called the *local time of collision* between $Y_k$ and $Y_{k+1}$. The vector of collision terms $L = (L_{(1,2)}, \ldots, L_{(N-1,N)})'$ is called the *vector of local times*.

Existence and uniqueness for systems of competing particles from Definition 16 is proved below. (This straightforward proof is completely analogous to the proof for competing Brownian particles, which was given in [71, subsection 2.1].)

We can also define *infinite* systems of competing particles. Chapter 7 of this thesis, which is based on the paper [101], deals with infinite systems of competing Brownian particles in detail. It uses a few facts from Chapter 4.

**Definition 18.** Let $X_1, X_2, \ldots : \mathbb{R}_+ \to \mathbb{R}$ be continuous functions with $X_1(0) \leq X_2(0) \leq \ldots$ Take *parameters of collision*: real numbers $q_n^+, q_n^-$, $n = 1, 2, \ldots$, which satisfy (3.5).

Consider continuous functions $Y_1, Y_2, \ldots : \mathbb{R}_+ \to \mathbb{R}$, $L_{(1,2)}, L_{(2,3)}, \ldots : \mathbb{R}_+ \to \mathbb{R}$ such that (i), (ii), (iii) and (iv) from Definition 16 are true, for $k = 1, 2, \ldots$. We let $L_{(0,1)} \equiv 0$, as in Definition 16. Then the system $Y = (Y_1, Y_2, \ldots)$ is called an *infinite system of competing particles* with *driving function* $X = (X_1, X_2, \ldots)$ and *parameters of collision* $(q_n^{\pm})_{n \geq 1}$. All other terms are defined as in Definition 16. Similarly, Definition 17 can be adapted for infinite number of Brownian particles.

Existence and uniqueness theorem is much harder to prove for infinite systems than for finite systems. Studying infinite systems of competing Brownian particles is the topic of Chapter 7 (which corresponds to [101]), where we prove, in particular, existence results.

In this chapter (see Remark 9), we state and prove a few comparison theorems for infinite systems, *assuming they exist*.

*Remark* 4. In the rest of the thesis, when we use the term *parameters of collision*, we always assume that they satisfy condition (3.5).

### 3.4 Systems of Competing Particles and the Skorohod Problem

The gap process for a system of competing particles is a solution to the Skorohod problem in the orthant. In particular, the gap process for competing Brownian particles is an SRBM in the orthant.

**Lemma 3.4.1.** *For a system of competing particles from Definition* 16*, its gap process is a solution to the Skorohod problem in the orthant* $\mathbb{R}_+^{N-1}$ *with reflection matrix*

$$R = \begin{bmatrix} 1 & -q_2^- & 0 & 0 & \dots & 0 & 0 \\ -q_2^+ & 1 & -q_3^- & 0 & \dots & 0 & 0 \\ 0 & -q_3^+ & 1 & -q_4^- & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & -q_{N-1}^- \\ 0 & 0 & 0 & 0 & \dots & -q_{N-1}^+ & 1 \end{bmatrix} \tag{3.6}$$

*and driving function*

$$(X_2 - X_1, X_3 - X_2, \dots, X_N - X_{N-1})' . \tag{3.7}$$

*Moreover, the matrix $R$ in* (3.6) *is a reflection nonsingular $\mathcal{M}$-matrix.*

*Proof.* Just use the property (i) from Definition 16; the gap process has the following representation:

$$Z_k(t) = Y_{k+1}(t) - Y_k(t) = X_{k+1}(t) - X_k(t) + L_{(k,k+1)}(t) - q_k^+ L_{(k-1,k)}(t) - q_{k+1}^- L_{(k+1,k+2)}(t),$$

for $k = 1, \dots, N-1$, $t \ge 0$. That $R$ is a reflection nonsingular $\mathcal{M}$-matrix is proved in [71]. For the sake of completeness, let us exhibit the proof. Let us show that $R$ is an inverse-positive matrix. Let $Q = I_{N-1} - R$. Note that $Q$ is a nonnegative irreducible matrix, all its column sums are less than or equal to 1, and the column sum for the first column strictly less than 1. Therefore, its spectral radius is strictly less than 1. The proof is in [71, Section 2.1] and [85, p.682]; see also [85, Exercise 8.3.7(b)]. Therefore, $R = I_{N-1} - Q$ is inverse-positive. Since, in addition, $r_{ij} \le 0$ for $i \ne j$, by Lemma 2.2.1 we have: $R$ is an $\mathcal{M}$-matrix. □

**Corollary 3.4.2.** *For a system of competing Brownian particles from* 16*, its gap process is an* $\mathrm{SRBM}^{N-1}(R, \mu, A)$*, where* $R$ *is given by* (3.6)*, and*

$$A = \begin{bmatrix} \sigma_1^2 + \sigma_2^2 & -\sigma_2^2 & 0 & 0 & \dots & 0 & 0 \\ -\sigma_2^2 & \sigma_2^2 + \sigma_3^2 & -\sigma_3^2 & 0 & \dots & 0 & 0 \\ 0 & -\sigma_3^2 & \sigma_3^2 + \sigma_4^2 & -\sigma_4^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \sigma_{N-2}^2 + \sigma_{N-1}^2 & -\sigma_{N-1}^2 \\ 0 & 0 & 0 & 0 & \dots & -\sigma_{N-1}^2 & \sigma_{N-1}^2 + \sigma_N^2 \end{bmatrix}, \tag{3.8}$$

$$\mu = (g_2 - g_1, g_3 - g_4, \dots, g_N - g_{N-1})'. \tag{3.9}$$

*Proof.* This follows directly from Lemma 3.4.1; it was, in fact, already proved in [71, Secton 2.1]. □

This connection allows us to prove existence and uniqueness for systems of competing particles.

**Lemma 3.4.3.** *Fix the number of particles* $N \ge 2$*. Also, fix parameters of collision* $q_1^{\pm}, \dots, q_N^{\pm}$*. For every continuous function* $X : \mathbb{R}_+ \to \mathbb{R}^N$ *with* $X(0) \in \mathcal{W}_N$*, there exists a unique system of competing particles with this driving function and parameters of collision.*

*Proof.* Consider the gaps between consecutive particles:

$$Z_k(t) := Y_{k+1}(t) - Y_k(t), \quad k = 1, \dots, N-1.$$

Note that the matrix $R$ from (3.6) is a reflection nonsingular $\mathcal{M}$-matrix, see Lemma 3.4.1, and the function (3.7) is continuous. Therefore, the solution to the Skorohod problem in

$\mathbb{R}_+^{N-1}$ with reflection matrix $R$ and driving function (3.7) exists and is unique. But this solution is the gap process, according to Lemma 3.4.1. Also, note that

$$\begin{cases} Y_1(t) = X_1(t) - q_1^- L_{(1,2)}(t), \\ Y_2(t) = X_2(t) + q_2^+ L_{(1,2)}(t) - q_2^- L_{(2,3)}(t), \\ \dots \\ Y_N(t) = X_N(t) - q_N^- L_{(N-1,N)}(t) \end{cases}$$

We can find a linear combination of $Y_1, \dots, Y_N$ which eliminates the collision terms: let

$$\alpha_1 = 1,\ \alpha_2 = \frac{q_1^-}{q_2^+},\ \alpha_3 = \frac{q_1^- q_2^-}{q_2^+ q_3^+}, \dots \tag{3.10}$$

then

$$Z_0(t) \equiv \alpha_1 X_1(t) + \dots + \alpha_N X_N(t) = \alpha_1 Y_1(t) + \dots + \alpha_N Y_N(t).$$

So we have constructed

$$Z_0(t) = \alpha_1 Y_1(t) + \dots + \alpha_N Y_N(t),\ Z_1 = Y_2 - Y_1, \dots, Z_{N-1} = Y_N - Y_{N-1}.$$

These functions $Z_0, Z_1, \dots, Z_{N-1}$ are unique. Now we can solve for $Y_1, \dots, Y_N$. Let

$$\tilde{Z} = (Z_0, \dots, Z_{N-1})' \in \mathbb{R}^N,$$

then $\tilde{Z}(t) = CY(t)$, where

$$C = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_N \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Then $Y(t) = C^{-1}\tilde{Z}(t)$ for $t \geq 0$. $\square$

### 3.5 The Gap Process for Competing Brownian Particles

The results of this subsection are taken from [3], [2], [57], [71], [53], [52], [124]. We can define the gap process for finite systems of competing Brownian particles (both classical and ranked) essentially in the same way as for the infinite Atlas model in the Introduction. For finite models, the gap process is finite-dimensional.

**Definition 19.** Consider a finite system (classical or ranked) of $N$ competing Brownian particles. Let

$$Z_k(t) = Y_{k+1}(t) - Y_k(t), \quad k = 1, \ldots, N-1, \quad t \ge 0.$$

Then the process $Z = (Z(t), t \ge 0)$, $Z(t) = (Z_1(t), \ldots, Z_{N-1}(t))'$ is called the *gap process*. The component $Z_k = (Z_k(t), t \ge 0)$ is called the *gap between the $k$th and $k+1$st ranked particles*.

The following propositions about the gap process are already known. We present them in a slightly different form then that from the sources cited above; for the sake of completeness, we present short outlines of their proofs. Let

$$R = \begin{bmatrix} 1 & -q_2^- & 0 & 0 & \ldots & 0 & 0 \\ -q_2^+ & 1 & -q_3^- & 0 & \ldots & 0 & 0 \\ 0 & -q_3^+ & 1 & -q_4^- & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & -q_{N-1}^- \\ 0 & 0 & 0 & 0 & \ldots & -q_{N-1}^+ & 1 \end{bmatrix}, \tag{3.11}$$

$$\mu = (g_2 - g_1, g_3 - g_4, \ldots, g_N - g_{N-1})'. \tag{3.12}$$

The following result is taken from [71], [52].

**Proposition 3.5.1.** *(i) The matrix $R$ is invertible, and $R^{-1} \ge 0$, with strictly positive diagonal elements $(R^{-1})_{kk}$, $k = 1, \ldots, N-1$.*

*(ii) The family of random variables $Z(t), t \ge 0$, is tight in $\mathbb{R}_+^{N-1}$, if and only if $R^{-1}\mu < 0$. In this case, for every initial distribution of $Y(0)$ we have: $Z(t) \Rightarrow \pi$ as $t \to \infty$, where $\pi$ is a unique stationary distribution of $Z$.*

*(iii) If, in addition, the* skew-symmetry condition *holds:*

$$(q_{k-1}^- + q_{k+1}^+)\sigma_k^2 = q_k^-\sigma_{k+1}^2 + q_k^+\sigma_{k-1}^2, \quad k = 2, \dots, N-1, \tag{3.13}$$

*then*

$$\pi = \bigotimes_{k=1}^{N-1} \mathcal{E}(\lambda_k), \quad \lambda_k = \frac{2}{\sigma_k^2 + \sigma_{k+1}^2}\left(-R^{-1}\mu\right)_k, \quad k = 1, \dots, N-1.$$

*Proof.* Part (i) was proved in [71, subsection 2.1]; see also [103, Lemma 2.1], which in this thesis corresponds to Lemma 2.2.1, with regard to the matrix $R$. Part (ii) of the statement follows from properties of an SRBM mentioned in Chapter 2, in particular, from Propositions 2.4.1, 2.4.2, 2.4.5 and Corollary 2.4.7. The *skew-symmetry condition* for an SRBM is written in the form

$$RD + DR' = 2A,$$

where $D = \mathrm{diag}(A)$ is the $(N-1) \times (N-1)$-diagonal matrix with the same diagonal entries as $A$. As mentioned in [125, Theorem 3.5], this is a necessary and sufficient condition for the stationary distribution to have product-of-exponentials form. This condition can be rewritten for $R$ and $A$ from (3.6) and (3.8) as (5.5). Take $i, j = 1, \dots, N-1$ and consider the condition

$$r_{ij}a_{jj} + r_{ji}a_{ii} = 2a_{ij}. \tag{3.14}$$

If $i = j$, then (3.14) is always true, because for such $i$, $j$ we have: $r_{ij} = r_{ji} = 1$, and $a_{ii} = a_{ij} = a_{jj} = \sigma_i^2 + \sigma_{i+1}^2$. If $|i-j| \ge 2$, then (3.14) is also always true, since $r_{ij} = r_{ji} = a_{ij} = 0$. Since the left-hand side and the right-hand side of (3.14) remain the same if we swap $i$ and $j$, we need only to check this condition for $j = k$, $i = k-1$, where $k = 2, \dots, N-1$. We get:

$$r_{ij} = -q_k^-, \quad r_{ji} = -q_k^+, \; a_{jj} = \sigma_k^2 + \sigma_{k+1}^2, \; a_{ii} = \sigma_{k-1}^2 + \sigma_k^2, \; a_{ij} = -\sigma_k^2.$$

Therefore, the condition (3.14) takes the form

$$-q_k^-\left(\sigma_k^2+\sigma_{k+1}^2\right)-q_k^+\left(\sigma_{k-1}^2+\sigma_k^2\right)=-2\sigma_k^2.$$

This is equivalent to

$$\left(2-q_k^--q_k^+\right)\sigma_k^2=q_k^-\sigma_{k+1}^2+q_k^+\sigma_{k-1}^2. \tag{3.15}$$

Note that $q_k^-+q_{k+1}^+=1$ and $q_k^++q_{k-1}^-=1$. Therefore, we can rewrite (3.15) as in (3.13) $\square$

For the case of symmetric collisions, we can refine Proposition 3.5.1. Let

$$\overline{g}_k:=\left(g_1+\ldots+g_k\right)/k \quad \text{for} \quad k=1,\ldots,N.$$

The following result is taken from [2], [3].

**Proposition 3.5.2.** *For the case of symmetric collisions $q_k^\pm=1/2,\ k=1,\ldots,N$,*

*(i) The vector $R^{-1}\mu$ can be represented as*

$$\begin{aligned}-R^{-1}\mu&=2\left(g_1-\overline{g}_N,g_1+g_2-2\overline{g}_N,\ldots,g_1+g_2+\ldots+g_{N-1}-(N-1)\overline{g}_N\right)'\\&=2\left(g_1-\overline{g}_N,2\left(\overline{g}_2-\overline{g}_N\right),\ldots,(N-1)\left(\overline{g}_{N-1}-\overline{g}_N\right)\right)'.\end{aligned}$$

*(ii) The tightness condition from Proposition* 3.5.1 *can be written as*

$$\overline{g}_k>\overline{g}_N,\quad k=1,\ldots,N-1.$$

*(iii) The skew-symmety condition can be equivalently written as*

$$\sigma_{k+1}^2-\sigma_k^2=\sigma_k^2-\sigma_{k-1}^2,\quad k=2,\ldots,N-1;$$

*in other words, $\sigma_k^2$ must linearly depend on $k$.*

*(iv) If both the tightness condition and the skew-symmetry condition are true, then*

$$\pi=\bigotimes_{k=1}^{N-1}\mathcal{E}(\lambda_k),\quad \lambda_k:=\frac{4k}{\sigma_k^2+\sigma_{k+1}^2}\left(\overline{g}_k-\overline{g}_N\right).$$

*Proof.* Let us show (i). It suffices to show that if

$$b = (g_1 - \overline{g}_N, g_1 + g_2 - 2\overline{g}_N, \ldots, g_1 + g_2 + \ldots + g_{N-1} - (N-1)\overline{g}_N)',$$

then

$$Rb = -\frac{1}{2}\mu = \frac{1}{2}(g_1 - g_2, g_2 - g_3, \ldots, g_{N-1} - g_N)'.$$

The matrix $R$ has the form (3.6) with $q_n^{\pm} = 1/2,\ n = 1, \ldots, N$, so

$$R = \begin{bmatrix} 1 & -1/2 & 0 & 0 & \ldots & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 & \ldots & 0 & 0 \\ 0 & -1/2 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \ldots & 1 & -1/2 \\ 0 & 0 & 0 & 0 & \ldots & -1/2 & 1 \end{bmatrix} \tag{3.16}$$

Therefore,

$$(Rb)_1 = b_1 - \frac{1}{2}b_2 = g_1 - \overline{g}_N - \frac{1}{2}(g_1 + g_2 - 2\overline{g}_N) = \frac{1}{2}g_1 - \frac{1}{2}g_2.$$

Similarly, for $k = 2, \ldots, N-2$,

$$\begin{aligned}(Rb)_k &= -\frac{1}{2}b_{k-1} + b_k - \frac{1}{2}b_{k+1} = -\frac{1}{2}(b_{k-1} - b_k) + \frac{1}{2}(b_k - b_{k+1}) \\ &= -\frac{1}{2}(-g_k + \overline{g}) + \frac{1}{2}(-g_{k+1} + \overline{g}) = -\frac{1}{2}(g_{k+1} - g_k) = (-\frac{1}{2}\mu)_k.\end{aligned}$$

In the same way, the same check can be done for $k = N-1$. This proves (i). Part (iii) is straightforward, because now $q_k^{\pm} = 1/2$ for all $k$; parts (ii) and (iv) follow from (i) and (iii). □

*Example* 1. If $g_1 = 1,\ g_2 = g_3 = \ldots = g_N = 0$, and $\sigma_1 = \sigma_2 = \ldots = \sigma_N = 1$ (finite Atlas model with $N$ particles), then

$$\pi = \bigotimes_{k=1}^{N-1} \mathcal{E}\left(2\frac{N-k}{N}\right).$$

The following is a technical lemma.

**Lemma 3.5.3.** *Take a finite system of competing Brownian particles (either classical or ranked). For every $t > 0$, the probability that there is a tie at time $t$ is zero.*

*Proof.* There is a tie for a system of competing Brownian particles at time $t > 0$ if and only if the gap process at time $t$ hits the boundary of the orthant $\mathbb{R}_+^{N-1}$. But the gap process is an SRBM. And an SRBM $Z = (Z(t), t \ge 0)$ in $\mathbb{R}_+^{N-1}$ has the property

$$\mathbf{P}(Z(t) \in \partial\mathbb{R}_+^{N-1}) = 0 \ \text{ for every } \ t > 0,$$

see [52, Section 7, Lemma 7]. □

## *3.6 Infinite Systems: Definitions and Known Facts*

In this section, we define infinite classical systems of competing Brownian particles. Infinite systems with asymmetric collisions are defined and constructed in Chapter 7; this is one of the new results in this thesis. Here, we state only results which are already known. For more details, we refer the reader to Chapter 7, which is a version of the author's article [101]. Chapter 7 also contains detailed proofs of existence and uniqueness statements from [105] and [59]. These proofs are not due to the author, but we felt that it might be a good idea to include them in this thesis for the sake of completeness.

Assume the usual setting: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbf{P})$, with the filtration satisfying the usual conditions.

Fix parameters $g_1, g_2, \ldots \in \mathbb{R}$ and $\sigma_1, \sigma_2, \ldots > 0$. We say that a sequence $(x_n)_{n\ge 1}$ of real numbers is *rankable* if there exists a one-to-one mapping (permutation) $\mathbf{p} : \{1, 2, 3, \ldots\} \to \{1, 2, 3, \ldots\}$ which *ranks the components of* $x$:

$$x_{\mathbf{p}(i)} \le x_{\mathbf{p}(j)} \ \text{ for } \ i, j = 1, 2, \ldots, \quad i < j.$$

As in the case of finite systems, we *resolve ties* (when $x_i = x_j$ for $i \ne j$) in the lexicographic order: we take a permutation $\mathbf{p}$ which ranks the components of $x$, and, in addition, if $i < j$ and $x_{\mathbf{p}(i)} = x_{\mathbf{p}(j)}$, then $\mathbf{p}(i) < \mathbf{p}(j)$. There exists a unique such permutation $\mathbf{p}$, which is called *the ranking permutation* and is denoted by $\mathbf{p}_x$. For example, if $x = (2, 2, 1, 4, 5, 6, 7, \ldots)'$, then $\mathbf{p}_x(1) = 3,\ \mathbf{p}_x(2) = 1,\ \mathbf{p}_x(3) = 2,\ \mathbf{p}_x(n) = n,\ n \ge 4$. Not all sequences of real numbers are rankable: for example, $x = (1, 1/2, 1/3, \ldots)'$ is not rankable.

**Definition 20.** Consider a $\mathbb{R}^\infty$-valued process

$$X = (X(t), t \ge 0),\ X(t) = (X_n(t))_{n \ge 1},$$

with continuous adapted components, such that for every $t \ge 0$, the sequence $X(t) = (X_n(t))_{n\ge1}$ is rankable. Let $\mathbf{p}_t$ be the ranking permutation of $X(t)$. Let $W_1, W_2, \ldots$ be i.i.d. standard $(\mathcal{F}_t)_{t\ge0}$-Brownian motions. Assume that the process $X$ satisfies an SDE

$$dX_i(t) = \sum_{k=1}^{\infty} 1(\mathbf{p}_t(k) = i)\left(g_k \mathrm{d}t + \sigma_k \mathrm{d}W_i(t)\right), \quad i = 1, 2, \ldots$$

Then the process $X$ is called an *infinite classical system of competing Brownian particles* with *drift coefficients* $(g_k)_{k\ge1}$ and *diffusion coefficients* $(\sigma_k^2)_{k\ge1}$. For each $i = 1, 2, \ldots$ the component $X_i = (X_i(t), t \ge 0)$ is called the *$i$th named particle*. If we define $Y_k(t) \equiv X_{\mathbf{p}_t(k)}(t)$ for $t \ge 0$ and $k = 1, 2, \ldots$, then the process $Y_k = (Y_k(t), t \ge 0)$ is called the *$k$th ranked particle*. The $\mathbb{R}_+^\infty$-valued process

$$Z = (Z(t), t \ge 0), \quad Z(t) = (Z_k(t))_{k\ge1},$$

defined by

$$Z_k(t) = Y_{k+1}(t) - Y_k(t), \quad k = 1, 2, \ldots,\ t \ge 0,$$

is called the *gap process*. If $X(0) = x \in \mathbb{R}^\infty$, then we say that the system $X$ *starts from $x$*.

In the papers [105] and [59], an existence and uniqueness result was proved. We do not cite it here, instead referring the reader to Chapter 7, where we state it in a slightly different and arguably more convenient form, as Theorem 7.2.1.

As mentioned in the Introduction (Chapter 1), the *infinite Atlas model* is a particular case of a infinite classical system of competing Brownian particles, with

$$g_1 = 1, \quad g_2 = g_3 = \ldots = 0, \quad \sigma_1 = \sigma_2 = \ldots = 1.$$

The following result was proved in [89]

**Proposition 3.6.1.** *There exists a version of the infinite Atlas model with*

$$Z(t) \sim \bigotimes_{k=1}^{\infty} \mathcal{E}(2) \quad \text{for all} \quad t \geq 0.$$

This is an example of a stationary distribution for an infinite system. Whether it is unique or not is an open question. This is in stark contrast with finite system, where the stationary distribution for the gap process, if it exists, is always unique. In Chapter 7, we reprove this result in a simpler way, as a corollary of more general results for general infinite systems (not just the infinite Atlas model).

For infinite systems, we define triple collisions in the same way as for finite systems. In the paper [59], the following result about triple collisions was proved.

**Proposition 3.6.2.** *Under assumptions of Theorem* 7.2.1*, if the sequence* $(0, \sigma_1^2, \sigma_2^2, \ldots)$ *is concave, then there are a.s. no triple collisions. If there are a.s. no triple collisions, then the sequence* $(\sigma_1^2, \sigma_2^2, \ldots)$ *is concave. A strong solution exists and is pathwise unique up to the first moment of a triple collision.*

In Chapter 7, we improve the result about triple collisions. It turns out that it is necessary and sufficient for the sequence $(\sigma_1^2, \sigma_2^2, \ldots)$ to be concave for absence of triple collisions.

## 3.7 Propagation of Chaos and McKean-Vlasov Equation

Consider a system of $N$ randomly moving and interactng particles (not necessarily competing Brownian particles). They are, of course, not necessarily independent. Now, let $N \to \infty$. If it happens that the limiting processes are i.i.d., this phenomenon is called *propagation of chaos.* When the number of particles was finite, they were dependent on each other, so there was some “order” in the system. But when the number of particles started increasing to infinity, then the “order” vanished, in the sense that each particle started moving independently of all other particles. This term is also applicable if we are talking about real-valued or finite-dimensional random variables, instead of random processes.

A McKean-Vlasov equation is a type of SDE where the drift and diffusion coefficients depend not only on the current position of the solution, but on the current *distribution* of the solution. Namely, let $X = (X(t), t \geq 0)$ be a real-valued stochastic process, and let $F_t(x) := \mathbf{P}(X(t) \leq x)$ be the cumulative distribution function of $X(t)$, for every $t \geq 0$. Take a standard Brownian motion $W = (W(t), t \geq 0)$, as well as functions $g, \sigma : [0,1] \to \mathbb{R}$. Consider the following equation:

$$\mathrm{d}X(t) = g(F_t(X(t)))\mathrm{d}t + \sigma(F_t(X(t)))\mathrm{d}W(t). \tag{3.17}$$

Under some additional technical assumptions, see [69] and [67], we can also write a PDE for the cumulative distribution function $F_t(x)$:

$$\frac{\partial F_t(x)}{\partial t} = \frac{1}{2}\frac{\partial^2(\Sigma(F_t(x)))}{\partial x^2} - \frac{\partial(G(F_t(x)))}{\partial x}, \tag{3.18}$$

where $G$ and $\Sigma$ are antiderivatives of $g$ and $\sigma^2$:

$$G(x) := \int_0^x g(y)\mathrm{d}y, \quad \Sigma(x) := \int_0^x \sigma^2(y)\mathrm{d}y.$$

This PDE (3.18) is called a *porous medium equation*. This equation describes various physical phenomena such as infiltration of water into a porous medium an evaporation of water from soil, see [117] and the references therein. This is a *quasilinear parabolic equation* with respect to the two-variable function $F_t(x)$, which means that it is linear in the derivatives of the function $F_t(x)$, but not in this function itself. In this respect, this process $X$ is different from an ordinary diffusion process, when the PDE for its cumulative distribution function is linear parabolic. This is why sometimes the process $X$ is called a *nonlinear diffusion process*.

The so-called *Vlasov equation* models plasma consisting of charged particles with Coulomb interaction. In [84] and [83], McKean observed that in a large ensemble of plasma particles, each individual particle moves randomly with cumulative distribution function as in (3.18). Note that the equation is the same for each individual particle; this is precisely the phenomenon of propagation of chaos. This equation (3.18) was used to describe limiting behavior for some other large systems of interacting particles in [19], [42], [41], [112] and [113].

Now, recall the classical system of $N$ competing Brownian particles from Definition 12:

$$\mathrm{d}X_i^{(N)}(t) = \sum_{k=1}^{N} (\mathbf{p}_t(k) = i) \left( g_k^{(N)} \mathrm{d}t + \sigma_k^{(N)} \mathrm{d}W_i(t) \right), \quad i = 1, \dots, N. \tag{3.19}$$

We explicitly stated dependence of $N$ in the superscript. Here, $\mathbf{p}_t$ is the *ranking permutation* on $\{1, \dots, N\}$: if the particle $X_i$ has rank $k$ at time $t$, then $\mathbf{p}_t(k) = i$, and $\mathbf{p}_t^{-1}(i) = k$. Now, consider an *empirical measure*:

$$\mu_t^{(N)} = \frac{1}{N} \sum_{k=1}^{N} \delta_{X_i(t)} = \frac{1}{N} \sum_{k=1}^{N} \delta_{X_{(k)}(t)},$$

with cumulative distribution function

$$F_t^{(N)}(x) := \frac{1}{N} \#\{i = 1, \dots, N \mid X_i(t) \le x\} = \frac{1}{N} \max\{k = 1, \dots, N \mid X_{(k)}(t) \le x\}.$$

Therefore, if there is no tie at time $t$ (which happens with probability 1)

$$F_t^{(N)}(X_i(t)) = \frac{k}{N} = \frac{p_t^{-1}(i)}{N}.$$

Define the functions

$$g^{(N)}, \sigma^{(N)} : \left\{ \frac{1}{N}, \dots, \frac{N-1}{N}, 1 \right\} \to \mathbb{R}$$

as follows:

$$g^{(N)} \left( \frac{k}{N} \right) := g_k^{(N)}, \quad \sigma^{(N)} \left( \frac{k}{N} \right) := \sigma_k^{(N)}.$$

We can write the SDE (3.19) as

$$\mathrm{d}X_i^{(N)}(t) = g^{(N)} \left( F_t^{(N)}(X_i(t)) \right) \mathrm{d}t + \sigma^{(N)} \left( F_t^{(N)}(X_i(t)) \right) \mathrm{d}W_i(t), \quad i = 1, \dots, N. \tag{3.20}$$

One can see that this bears clear resemblance to the SDE (3.17) for a nonlinear process. The equation (3.19), or its equivalent formulation (3.20), can be viewed as a discrete version of the McKean-Vlasov equation (3.17). So it is natural to anticipate that there is a special version of the law of large numbers. If, in some sense, $g^{(N)}$ and $\sigma^{(N)}$ are discrete versions of the functions $g$ and $\sigma$, then the discrete system (3.20) converges to a continuous system (3.17). This means that the empirical measure $\mu_t^{(N)}$ converges to the distribution of $X(t)$ from (3.17).

This was proved in a general version in [20] as a trivial corollary of a large deviations result. (As always, large deviations principle serves as a refinement of a law of large numbers, and the latter trivially follows from the former.) See also Section 3.9, devoted to literature review.

### 3.8 Applications to Stochastic Portfolio Theory

Let us outline a brief and informal introduction to *Stochastic Portfolio Theory*, a newly developed area of financial mathematics. The foundations of this theory were developed in the articles [33], [28], [25], [26], and the mongraph [27]. This theory is descriptive, as opposed to normative; it is consistent with the actual real-world stock market behavior and allows to construct successful investment strategies.

Let us model a market with $N$ stocks. Consider $N$ strictly positive stochastic processes

$$X_k = (X_k(t), t \ge 0), \quad k = 1, \ldots, N.$$

The quantity $X_k(t)$ is the capitalization of $k$th stock at time $t$. The *total market capitalization* is defined as

$$S(t) := X_1(t) + \ldots + X_N(t).$$

The *market weight* of the $k$th stock is given by

$$\mu_k(t) = \frac{X_k(t)}{S(t)}.$$

A *portfolio* is an $\mathbb{R}^N$-valued process

$$\pi = (\pi(t), t \ge 0), \quad \pi(t) = (\pi_1(t), \ldots, \pi_N(t))',$$

with $|\pi_i(t)| \le K_\pi$ and$\pi_1(t) + \ldots + \pi_N(t) = 1$. The quantity $\pi_k(t)$ represents a share of the total wealth invested in the $k$th stock. This framework allows short selling, when some $\pi_k(t)$ are negative. If all $\pi_k(t)$ are nonnegative, this is called a *long-only portfolio*. The *wealth process* $V^\pi = (V^\pi(t), t \ge 0)$ is a strictly positive process such that $V^\pi(0) = 1$ and

$$\frac{\mathrm{d}V^\pi(t)}{V^\pi(t)} = \sum_{k=1}^N \pi_k(t) \frac{\mathrm{d}X_k(t)}{X_k(t)}, \quad t \ge 0.$$

One example of a portfolio is the *market portfolio*, where $\pi = \mu$. This means that we simply buy a share of the whole stock market. The corresponding wealth process is $V^\mu(t) = S(t)/S(0)$. We say that portfolio $\pi$ represents an *arbitrage opportunity relative to portfolio $\rho$ on the time horizon $T$* if

$$V^\pi(T) > V^\rho(T) \ \text{ a.s.,} \ \ V^\pi(T) > V^\rho(T) \ \text{ with positive probability.}$$

If the strict inequality holds a.s. we say that $\pi$ provides a *strong relative arbitrage*. The market is called *diverse* if there exists $\delta > 0$ such that

$$\mu_k(t) \le 1 - \delta \ \text{ for all } \ k = 1, \ldots, N \ \text{ and } \ t \ge 0.$$

Examples of such models were constructed in [32], [31] and [99]. We say that a model has *sufficient intrinsic volatility* if there exists $\lambda > 0$ such that for all $t \ge 0$ and $\xi = (\xi_1, \ldots, \xi_N)' \in \mathbb{R}^N$,

$$\sum_{k=1}^N \sum_{l=1}^N \xi_k \xi_l \frac{d\langle \log X_k(t), \log X_l(t)\rangle_t}{dt} \ge \lambda \|\xi\|^2.$$

The following fundamental theorem was proved in [27], [31], [32].

**Proposition 3.8.1.** *For a diverse market model with sufficient intrinsic volatility, there exists a portfolio which provides a strong relative arbitrage relative to the market over sufficiently long-term horizon $T$.*

One example is a *diversity-weighted portfolio*: take some $p \in (0, 1)$, and let

$$\pi_k(t) = \frac{\mu_k^p(t)}{\mu_1^p(t) + \ldots + \mu_N^p(t)}, \quad t \ge 0, \quad k = 1, \ldots, N.$$

See also a recent paper [118]. More examples can be constructed using *functionally generated portfolios*, see [27, Chapter 3].

We say that a market model *admits an equivalent martingale measure* $\mathbb{Q}$ if $\mathbb{Q}$ is equivalent to the original measure $\mathbf{P}$ on the filtration $\mathcal{F}_T$ for each $T > 0$, and under this new measure, each process $X_i$, $i = 1, \ldots, N$, is a martingale.

It can be shown that if a model admits an equivalent martingale measure, then it does not allow arbitrage (relative to any portfolio). Let us quote the book [27, Section 3.3]:

> It is difficult, if not impossible, to test the validity of the no-arbitrage hypothesis empirically. In the literature, no-arbitrage frequently follows from the assumed existence of an equivalent martingale measure, and the existence of such a measure is not amenable to statistical verification. [...] [The above example] shows that arbitrage is possible in a market that seems eminently well-behaved. [...] From a normative point of view, weak diversity seems like an innocuous enough assumption, and it would surely be imposed upon an actual equity market by any credible antitrust regulation. Compare this mild assumption to the all-encompassing existence of an equivalent martingale measure. The former implies arbitrage, the latter no-arbitrage. [...] In light of this discussion, it would seem that the no-arbitrage hypothesis must be relegated to the class of "empirically undecidable" statements, along with the older problem of determining the number of angels that can dance on the head of a pin.

One class of market models is based on competing Brownian particles. Take a classical system of $N$ competing Brownian particles $Y = (Y_1, \ldots, Y_N)'$ with drift coefficients $(g_n)_{1\le n\le N}$ and diffusion coefficients $(\sigma_k^2)_{1\le k\le N}$. Now, let

$$X_k(t) := e^{Y_k(t)}, \quad t \ge 0, \quad k = 1, \ldots, N.$$

Then $X = (X_1, \ldots, X_N)'$ is a market model. This model was introduced in [3]. It is not diverse and it does not allow arbitrage, because it admits an equivalent martingale measure (Girsanov removal of drifts).

One of the aims of this model is to capture the real-world phenomenon which was already mentioned in the Introduction: stocks with smaller capitalizations have larger growth rates and larger volatilities. In the context of this model, we must have

$$g_1 > g_2 > \ldots > g_N \ \text{ and } \ \sigma_1^2 > \sigma_2^2 > \ldots > \sigma_N^2.$$

There is another usage of this model: to explain the Fernholz curve.

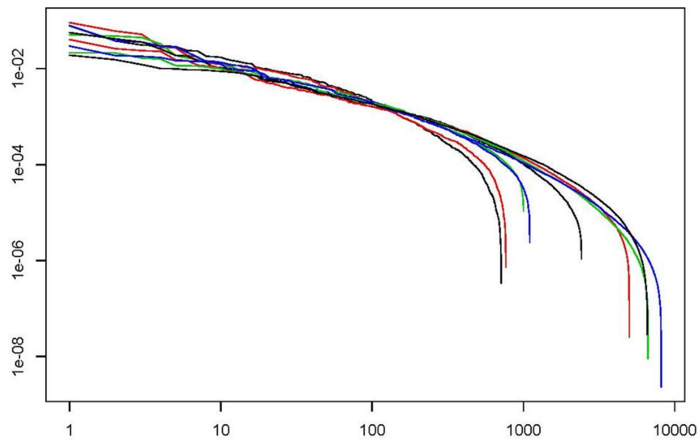


Figure 3.1: Capital distribution curves: 1929-1999

Take real-world data of stocks. Calculate their market weights and rank them from top to bottom according to their capitalizations (or, equivalently, their market weights): let $\mu_{(k)}(t)$ be the $k$th largest market weight at time $t$. Consider the log-log plot $\log k \mapsto \log \mu_{(k)}(t)$, at different moments $t$. For this example, take $t$ to be December 31 of eight different years: 1929, 1939, 1949, 1959, 1969, 1979, 1989, 1999. (More detailed information on which stocks were included can be found in [27, Section 5.1].) The result is shown in FIGURE 3.8.

The plot shows remarkable stability over time and linearity in its upper part.

We can explain this with the use of the model described above, based on competing Brownian particles. More precisely: Take large $N$ and consider a system of competing Brownian particles with drift coefficients $g_1, \ldots, g_N$ and diffusion coefficients $\sigma_1^2, \ldots, \sigma_N^2$. Assume that the gap process is in its stationary distribution. The ranked market weights are functions of the gap process. Indeed, if $Y_{(j)}(t)$ is the $j$th *smallest* particle at time $t$ (note the difference between ranking the market weights and ranking competing Brownian particles), then

$$\mu_{(k)}(t) = \frac{\exp\left(Y_{(N-k+1)}(t)\right)}{\sum\limits_{j=1}^{N}\exp\left(Y_{(j)}(t)\right)} = \frac{1}{\sum\limits_{j=1}^{N}\exp\left(Y_{(j)}(t) - Y_{(N-k+1)}(t)\right)},$$

and $Y_{(j)}(t) - Y_{(N-k+1)}(t)$ is a sum of a few gaps. So the vector of ranked market weights

$$\left(\mu_{(1)}(t), \ldots, \mu_{(N)}(t)\right)'$$

is also in its stationary distribution. We can take large $N$ and adjust parameters $g_1, \ldots, g_N, \sigma_1^2, \ldots, \sigma_N^2$, so that this log-log plot under the stationary distribution has the form shown in Figure 3.8. This was done in [3]. In the paper [11], this stationary distribution is investigated when $N \to \infty$, under some assumptions on the coefficients. It turns out that in some cases (for example, the Atlas model), as $N \to \infty$, the stationary distribution for ranked market weights converges to the so-called *Poisson-Dirichlet point process*, which has the property that the log-log plot is (approximately) linear.

### *3.9 Literature Review*

Classical systems from Definition 12 were introduced in [3]. The formula (3.2), together with the connection between the gap process and an SRBM in the orthant, were proved in [2]. Proposition 3.5.2 was proved in [89], [2] and the thesis [57] by Tomoyuki Ichiba. The paper [11] contains limit theorems for the stationary distribution of the gap process as the number $N$ of particles goes to infinity. Rate of convergence for the gap process to this stationary distribution is found in [60], [65]; in the latter paper, the rate of convergence in $\chi^2$-distance does not depend on $N$, the number of particles. Concentration of measure results are proved in [90]. Poincare inequalities for the system in its stationary distribution are proved in [65] and [60]. The paper [68] deals with a *small noise limit*, when diffusion coefficients tends to zero.

Relation to stochastic finance is shown in the articles [66], [11], in the survey [31], and in an earlier book [27]. See also a recent article [34], which uses the model to study economic inequality and tax policy.

Propagation of chaos (see Section 3.7) was studied in the paper [67]. The paper [65] also contains some comparison results about classical systems of competing Brownian particles, somewhat similar to the ones in Chapter 4. The paper [20] deals with large deviations for classical systems of competing Brownian particles.

There are several generalizations of these systems: [105] (systems of competing Levy particles), [30], [29], [2] (*second-order stock market models*, when drift and diffusion coefficients depend on both ranks and names).

As mentioned before, systems with asymmetric collisions from Definition 14 were introduced in [71]; this paper also deals with triple collisions (obtaining a partial result on the problem which is completely resolved in Chapter 5 of the current thesis), as well as recurrence and stationary distributions of the gap process.

Infinite (classical) systems of competing Brownian particles were introduced in [89] (where they proved that the distribution $\pi_\infty$ from (1.7), which is an infinite product of exponential distributions with rates 2 is a stationary distribution for the gap process of the infinite Atlas model, see Chapter 1). The papers [105] and [59] deal with existence and uniqueness questions, as well as triple collisions. Sections 5 and 7 are continuation of research carried out in these three papers. In a recent paper [21], it is proved that the scaling limit of the lowest-ranked particle in the infinite Atlas model is the fractional Brownian motion with Hurst parameter $H = 1/4$. This is similar to the Harris model from [45], which is a double-sided infinite system of Brownian particles with zero drifts and unit diffusions (we consider double-sided infinite systems in Chapter 8 of this thesis).

Other ordered particle systems derived from independent driftless Brownian motions were studied by Arratia in [1], and by Sznitman in [112] and [113]. Several other papers study connections between systems of queues and one-dimensional interacting particle systems: [79], [109], [35], [36], [37], [104]. Links to the directed percolation and the directed polymer models, as well as the GUE random matrix ensemble, can be found in [5] and [87]. (References in this paragraph are quoted from [89].)

Systems of competing Brownian particles with asymmetric collisions are related to the theory of exclusion processes: it was proved in [71, Section 3] that these systems are scaling limits of asymmetrically colliding random walks, which constitute a certain type of exclusion processes. In addition, thse systems are also related to random matrices and random surfaces evolving according to the KPZ equation, see [38].

Propagation of chaos results and convergence to McKean-Vlasov equations were a subject of extensive research. This convergence was proved in [106] for the case when the system of competing Brownian particles from (3.19) has the gap process in its stationary distribution, and the function $\sigma^2$ is affine; in this case, this stationary distribution has product of exponentials form, see Proposition 3.5.2. It was proved in a stronger form (pathwise rather than weak pointwise convergence) in [65] for the case when $\sigma^2$ is constant, and $G$ is convex. In this article, they also studied propagation of chaos for *projected particle system*, that is, the projection of $(X_1, \ldots, X_N)'$ onto the hyperplane $z_1 + \ldots + z_N = 0$. In [67], convergence of the empirical cumulative distribution function to (3.18), and convergence of the empirical measure to the solution of (3.17) was shown under fairly weak conditions on $g$ and $\sigma^2$. In fact, in this paper, systems of competing Brownian particles were used as discrete approximation to show existence of the solution to (3.17). In the papers [65] and [67], a system of competing Brownian particles need not have the gap process in its stationary distribution. In the paper [64], they proved propagation of chaos for a particular case: $g(u) := 0$ and $si^2(u) := 2qu^{q-1}$ for some $q > 1$. See also a related paper [63].

In the paper [98], propagation of chaos is established for stationary distributions instead of the processes. Namely, it is shown that a stationary distribution for a projected system of competing Brownian particles converges to the stationary distribution for the McKean-Vlasov equation.

# Chapter 4

# COMPARISON TECHNIQUES

This chapter, which corresponds to the author's paper [100], is organized as follows. In Section 4.1, we provide some intuitive simple examples. In Section 4.2, we state the main results: Theorems 4.2.1 and 4.2.2. Section 4.3 is devoted to simple corollaries, which are applied in later chapters. Section 4.4 contains proofs of Theorems 4.2.1 and 4.2.2. In Section 4.5, we study the case of totally asymmetric collisions, when parameters of collision are allowed to be equal to 0 or 1. Section 4.6 is an Appendix, which contains some technical lemmata.

## *4.1 Simple Examples*

As a preview, let us mention a few (rather intuitive) results proved in this chapter. They are applied in Chapter 7, which corresponds to the author's paper [101]. See also the sketch of the proof of Theorem 1.4.1 in the Introduction.

(i) If we remove a few competing Brownian particles $Y_{M+1}, \ldots, Y_N$ from the right, the positions of the remaining particles $Y_1(t), \ldots, Y_M(t)$ at any time $t \ge 0$ shift to the right (in the sense of stochastic comparison), because they no longer feel pressure from the right, exerted by the removed particles. Moreover, the local times $L_{(k,k+1)}(t)$ stochastically decrease, and the gaps $Z_k(t)$ stochastically increase, for $k = 1, \ldots, N-1$. (Corollary 4.3.8.)

(ii) If we shift (in the sense of stochastic comparison) initial positions $Y_k(0)$, $k = 1, \ldots, N$, of all competing Brownian particles to the right, then their positions $Y_k(t)$, at any fixed time $t \ge 0$ also shift to the right, in the sense of stochastic comparison. (Corollary 4.3.10 (i).)

(iii) If we stochastically increase the initial gaps $Z_k(0)$, $k = 1, \ldots, N-1$, between particles, then at any time $t \ge 0$ the values of the gaps $Z_k(t)$ also stochastically increase, and the local times $Y_{(k,k+1)}(t)$ stochastically decrease, for $k = 1, \dots, N-1$.(Corollary 4.3.10 (ii).)

(iv) If we stochastically increase the values of parameters $q_1^+, \dots, q_N^+$, then the particles $Y_1(t), \dots, Y_N(t)$ stochastically shift to the right. (Corollary 4.3.11.)

We get these (and similar) results as corollaries of the two main results stated in section 4.3: Theorems 4.2.1 and 4.2.2. These two theorems deal with general *systems of competing particles*, which are generalizations of competing Brownian particles: they have arbitrary continuous driving functions $\mathcal{X}_1(t), \dots, \mathcal{X}_N(t)$, in place of Brownian motions $g_1 t + \sigma_1 B_1(t), \dots, g_N t + \sigma_N B_N(t)$.

Although these results are intuitive and natural, their proofs turn out to be very complicated and technical. Essentially, we approximate the $\mathbb{R}^N$-valued function

$$(g_1 t + \sigma_1 B_1(t), \dots, g_N t + \sigma_N B_N(t))'$$

by piecewise linear functions with each piece parallel to a coordinate axis. For such piecewise linear functions, we can solve for $\mathcal{Y}_1, \dots, \mathcal{Y}_N$ explicitly and compare these solutions piece by piece.

## 4.2 Main Results: Theorems 4.2.1 and 4.2.2

Let us now state the two main results of this chapter. The first result is devoted to the Skorohod problem in the orthant. It states that the solution to the Skorohod problem and the boundary terms are, in some sense, monotone with respect to the driving function and the reflection matrix.

**Theorem 4.2.1.** *Fix the dimension $d \ge 1$ and let $S = \mathbb{R}^d_+$. Consider two continuous functions $X, \overline{X} : \mathbb{R}_+ \to \mathbb{R}^d$ such that $X(0),\ \overline{X}(0) \in S$, and*

$$X(0) \le \overline{X}(0), \quad X(t) - X(s) \le \overline{X}(t) - \overline{X}(s),\ t \ge s \ge 0. \tag{4.1}$$

*Take two $d \times d$ reflection nonsingular $\mathcal{M}$-matrices $R$ and $\overline{R}$ such that $R \le \overline{R}$. Let $Z$ and $\overline{Z}$ be the solutions to the Skorohod problems in the orthant $S$ with reflection matrices $R$, $\overline{R}$,* *and driving functions $X$, $\overline{X}$, respectively. Let $L$, $\overline{L}$ be the corresponding vectors of boundary terms. Then*

$$Z(t) \leq \overline{Z}(t), \quad L(t) - L(s) \geq \overline{L}(t) - \overline{L}(s), \quad t \geq s \geq 0.$$

Let us explain this informally. Suppose we make the values $X(t), t \geq 0$, and increments $X(t) - X(s),\ 0 \leq s \leq t$, of the driving function $X$, as well as the off-diagonal elements $r_{ij},\ i \neq j$, of the reflection matrix $R$, smaller. (The diagonal elements $r_{ii}$, $i = 1, \ldots, d$, by definition, are always equal to 1.) Then the value $Z(t)$ to the Skorohod problem $Z$ decreases (at any fixed time $t \geq 0$), and the values of boundary terms $L_i(t)$, $i = 1, \ldots, d$, increase.

This is what one would expect: if the driving function $X$ decreases, this will cause the "driven function" $Z$ also to decrease, at least until $Z$ is moving inside the orthant $S$. Indeed, $Z$ "wants to follow" $X$, by definition of the Skorohod problem. However, since the values $Z(t)$ of the function $Z$ become smaller at any fixed time $t \geq 0$, the process $Z$ hits the boundary more often.

And this leads to increase in the boundary terms, which grow when $Z$ hits the boundary, and which are "helping" $Z$ to stay in the orthant $S$. (Recall that the driving function $X$ starts from the orthant but can leave it later.) The boundary terms $L_j(t) \geq 0$ become larger, while the off-diagonal elements $r_{ij} \leq 0,\ i \neq j$, of the reflection matrix $R$ become smaller. So the terms $r_{ij}L_j(t) \leq 0$ become smaller for all $i \neq j$. The term $r_{ii}L_i(t) = L_i(t)$ is the only term in decomposition

$$Z_i = X_i + \sum_{j=1}^{d} r_{ij}L_j(t) \tag{4.2}$$

that becomes larger, but it cannot make $Z_i$ larger than it already is, because it grows only when $Z_i = 0$, and $Z_i \geq 0$ always.

*Remark* 5. Note that the condition that the reflection matrix $R$ has non-positive off-diagonal elements (in other words, that it is a nonsingular reflection $\mathcal{M}$-matrix) is crucial. Suppose that $r_{21} > 0$. When $Z$ hits the face $S_1$, that is, when $Z_1(t) = 0$, the boundary term $L_1$ might increase by some increment $\mathrm{d}L_1(t)$. So the component $Z_2$ might get additional increase

$r_{21}\mathrm{d}L_1(t)$. Consider a concrete example: two driving functions $X$ and $\overline{X}$, with

$$X_1(t) = -t,\ \overline{X}_1(t) = 1 - t,\ X_i(t) = \overline{X}_i(t) = 1,\ i = 2, \ldots, d.$$

These functions satisfy the conditions of Theorem 4.2.1. Let $R = \overline{R}$ be a reflection nonsingular $\mathcal{M}$-matrix. Let us solve the Skorohod problem in the orthant $S$ for reflection matrix $R$ and driving functions $X$ and $\overline{X}$. The function $X$ hits $S_1$ already at time $t = 0$, but $\overline{X}$ does this at time $t = 1$. So $Z_2$ gets some of this increase mentioned above before $\overline{Z}_2$ does. Actually, one can find the solutions explicitly: for $t \in [0, 1]$,

$$Z_2(t) = 1 + r_{21}t,\ \overline{Z}_2(t) = 1.$$

Therefore, the statement of Theorem 4.2.1 is not true in this case.

The part of Theorem 4.2.1 concerning the functions $Z$ and $\overline{Z}$ is already known: see [76], [93], [78], [44]. However, we present a different method of proof, which allows us to compare not just solutions to the Skorohod problem, but boundary terms as well. This comparison of boundary terms plays crucial role in some of the proofs in Chapter 7 (based on the author's paper [101]). We could not find the results about boundary terms in the existing literature; this served as a motivation for Theorem 4.2.1.

The other theorem deals with systems of competing particles. Consider a system of $N$ competing particles. If we increase the values and increments of driving functions, as well as the coefficients $q_n^+,\ n = 2, \ldots, N$, then the output $Y(t)$ (positions of competing particles) will increase, too. Increasing coefficients $q_n^+,\ n = 2, \ldots, N$, has the following sense: for each $n$, at every collision between the ranked particles $Y_n$ and $Y_{n+1}$, the share of the push going to $Y_{n+1}$ (which pushes this particle to the right) increases, and the share of the push going to $Y_n$ (which pushes this particle to the left) decreases.

**Theorem 4.2.2.** *Fix $N \geq 2$, the number of particles. Consider two continuous functions $X,\ \overline{X} : \mathbb{R}_+ \to \mathbb{R}^N$, with $X(0), \overline{X}(0) \in \mathcal{W}_N$, such that*

$$X(0) \leq \overline{X}(0),\ X(t) - X(s) \leq \overline{X}(t) - \overline{X}(s),\ 0 \leq s \leq t.$$

*Fix parameters of collision* $(q_n^{\pm})_{1\le n\le N}$ *and* $(\overline{q}_n^{\pm})_{1\le n\le N}$*, such that*

$$q_n^+ \le \overline{q}_n^+,\ n = 2, \ldots, N.$$

*Consider systems* $Y$ *and* $\overline{Y}$ *of competing particles with driving functions* $X$ *and* $\overline{X}$*, and parameters of collision* $(q_n^{\pm})_{1\le n\le N}$ *and* $(\overline{q}_n^{\pm})_{1\le n\le N}$*. Then*

$$Y(t) \le \overline{Y}(t),\ t \ge 0.$$

## 4.3 Corollaries

There are many corollaries of these two main results, which are straightforward but interesting. They are used in Chapter 7 (corresponding to [101]). We shall state and prove them in this subsection.

**Corollary 4.3.1.** *Take a* $d\times d$*-reflection nonsingular* $\mathcal{M}$*-matrix* $R$*. Consider two copies of an* $\mathrm{SRBM}^d(R, \mu, A)$*:* $Z$ *and* $\overline{Z}$*, starting from* $Z(0)$ *and* $\overline{Z}(0)$ *such that* $Z(0) \preceq \overline{Z}(0)$*. Let* $L$ *and* $\overline{L}$ *be the corresponding vectors of boundary terms. Then*

$$Z(t) \preceq \overline{Z}(t),\ t \ge 0;$$

$$L(t) - L(s) \succeq \overline{L}(t) - \overline{L}(s),\ 0 \le s \le t.$$

*Proof.* We can switch from stochastic domination $Z(0) \preceq \overline{Z}(0)$ to a.s. domination, by changing the probability space. Assume that $B = (B(t), t \ge 0)$ is a $d$-dimensional Brownian motion, starting at the origin, with drift vector $\mu$ and reflection matrix $A$. Then $Z$ and $\overline{Z}$ are solutions to the Skorohod problem in $\mathbb{R}^d_+$ with driving functions $Z(0) + B(t)$, $\overline{Z}(0) + B(t)$, respectively, and reflection matrix $R$, and $L$, $\overline{L}$ are corresponding vectors of boundary terms. The rest follows from Theorem 4.2.1. □

**Corollary 4.3.2.** *Fix* $N \ge 2$*, the number of particles. Also, fix parameters of collision* $(q_n^{\pm})_{1\le n\le N}$*. Take two continuous functions* $X, \overline{X} : \mathbb{R}_+ \to \mathbb{R}^N$ *such that for*

$$W = (X_2 - X_1, \ldots, X_N - X_{N-1})',\ \overline{W} = (\overline{X}_2 - \overline{X}_1, \ldots, \overline{X}_N - \overline{X}_{N-1})', \tag{4.3}$$

*we have:*

$$W(0) \le \overline{W}(0),\ W(t) - W(s) \le \overline{W}(t) - \overline{W}(s),\ 0 \le s \le t.$$

*Let $Y$, $\overline{Y}$ be the systems of competing particles with parameters of collision $(q_n^{\pm})_{1 \le n \le N}$ and driving functions $X$ and $\overline{X}$, respectively. Let $Z$, $\overline{Z}$ be the corresponding gap processes, and let $L$, $\overline{L}$ be the corresponding vectors of collision terms. Then*

$$Z(t) \le \overline{Z}(t),\ t \ge 0;\ L(t) - L(s) \ge \overline{L}(t) - \overline{L}(s),\ 0 \le s \le t.$$

*Proof.* The functions $Z$ and $\overline{Z}$ are solutions to the Skorohod problem in the orthant $\mathbb{R}_+^{N-1}$ with reflection matrix $R$ from (3.6) and driving functions $W$ and $\overline{W}$, respectively. The functions $L$ and $\overline{L}$ are the corresponding vectors of boundary terms for these two Skorohod problems. Apply Theorem 4.2.1 and finish the proof. □

**Corollary 4.3.3.** *Suppose $X : \mathbb{R}_+ \to \mathbb{R}^d$ is a continuous function with $X(0) \in S$. Fix a $d \times d$-reflection nonsingular $\mathcal{M}$-matrix $R$. Take a nonempty subset $I \subseteq \{1, \dots, d\}$ with $|I| = p$. Let $Z$ be the solution to the Skorohod problem in $S$ with reflection matrix $R$ and driving function $X$, and let $L$ be the corresponding vector of boundary terms. Also, let $\overline{Z}$ be the solution to the Skorohod problem in $\mathbb{R}_+^p$ with reflection matrix $[R]_I$ and driving function $[X]_I$, and let $\overline{L}$ be the corresponding vector of boundary terms. Then*

$$[Z(t)]_I \le \overline{Z}(t),\ t \ge 0;\ [L(t)]_I - [L(s)]_I \ge \overline{L}(t) - \overline{L}(s),\ 0 \le s \le t.$$

*Remark* 6. Corollary 4.3.3 has the following intuitive sense: suppose we remove a few components of the driving function. Then these (no longer existing) components do not hit zero and do not contribute (via boundary terms) to the decrease of the remaining components. If the component $j$ was removed but the component $i$ stayed, then in the equation (4.2) $Z_i(t)$ no longer has the term $r_{ij}L_j(t) \le 0$. Thus, $Z_i(t)$ becomes larger.

*Proof.* Recall that $Z(t) \equiv X(t) + RL(t)$. For $i \in I$, $t \ge 0$,

$$Z_i(t) = X_i(t) + \sum_{j\in I} r_{ij}L_j(t) + \sum_{j\notin I} r_{ij}L_j(t).$$

Therefore, $[Z]_I$ is the solution of the Skorohod problem in $\mathbb{R}^p_+$ with reflection matrix $[R]_I$ and driving function

$$\overline{X} = (\overline{X}_i)_{i\in I}, \quad \overline{X}_i(t) = X_i(t) + \sum_{j\notin I} r_{ij}L_j(t), \quad i \in I.$$

But $r_{ij} \le 0$ for $i \in I,\ j \in I^c$, because $R$ is a $\mathcal{Z}$-matrix. Moreover, each of the processes $L_j,\ j \in I^c$, is nondecreasing. Therefore,

$$\overline{X}_i(t) - \overline{X}_i(s) \le X_i(t) - X_i(s), \quad 0 \le s \le t, \quad i \in I.$$

Apply Theorem 4.2.1 and finish the proof. $\square$

The following corollary is a consequence (and a Brownian counterpart) of Corollary 4.3.3.

**Corollary 4.3.4.** *Take a $d\times d$ reflection nonsingular $\mathcal{M}$-matrix $R$, a $d\times d$ positive definite symmetric matrix $A$, and a drift vector $\mu \in \mathbb{R}^d$. Fix a nonempty subset $I \subseteq \{1,\dots,d\}$. Let*

$$Z = \mathrm{SRBM}^d(R,\mu,A), \quad \overline{Z} = \mathrm{SRBM}^{|I|}([R]_I,[\mu]_I,[A]_I)$$

*such that $[Z(0)]_I$ has the same law as $\overline{Z}(0)$. Then $[Z]_I \preceq \overline{Z}$.*

**Proposition 4.3.5.** *Take two $d\times d$ reflection nonsingular $\mathcal{M}$-matrices $R, \overline{R}$ such that $\overline{R} \le R$. Fix a vector $\mu \in \mathbb{R}^d$ and a positive definite symmetric $d\times d$-matrix $A$. Let*

$$Z = \mathrm{SRBM}^d(R,\mu,A), \quad \overline{Z} = \mathrm{SRBM}^d(\overline{R},\mu,A), \quad \textit{such that} \ \ Z(0) \preceq \overline{Z}(0).$$

*Then $Z \preceq \overline{Z}$.*

**Corollary 4.3.6.** *Let $1 < N \le M$. Fix a continuous function $X : \mathbb{R}_+ \to \mathbb{R}^M$ with $X(0) \in \mathcal{W}_M$. Fix parameters of collision $(q^{\pm}_n)_{1\le n\le M}$. Let $Y$ be the system of $M$ competing particles with parameters of collision $(q^{\pm}_n)_{1\le n\le M}$ and driving function $X$. Let $\overline{Y}$ be the system of $N$*

*competing particles with parameters of collision $(q_n^{\pm})_{1\le n\le N}$ and driving function $[X]_N$. Let $Z$, $\overline{Z}$ be the corresponding gap processes, and let $L$, $\overline{L}$ be the corresponding vectors of boundary terms. Then*

$$Z_k(t) \le \overline{Z}_k(t),\ k = 1, \dots, N-1,\ t \ge 0; \tag{4.4}$$

$$L_k(t) - L_k(s) \ge \overline{L}_k(t) - \overline{L}_k(s),\ k = 1, \dots, N-1,\ 0 \le s \le t; \tag{4.5}$$

$$Y_k(t) \le \overline{Y}_k(t),\ k = 1, \dots, N,\ t \ge 0. \tag{4.6}$$

*Remark* 7. Corollary 4.3.6 has the following meaning: if we take a system of competing particles and remove a few particles from the right, then there is "less pressure" on the remaining left particles which would push them further to the left. Therefore, the gaps become wider; there are less collisions, so the collision terms become smaller; and the remaining particles themselves shift to the right.

*Proof.* For the system $Y$, we can write the first $N$ particles as

$$\begin{cases} Y_1(t) = X_1(t) - q_1^- L_{(1,2)}(t), \\ Y_2(t) = X_2(t) + q_2^+ L_{(1,2)}(t) - q_2^- L_{(2,3)}(t), \\ \dots \\ Y_N(t) = X_N(t) + q_N^+ L_{(N-1,N)}(t) - q_N^- L_{(N,N+1)}(t). \end{cases}$$

So the vector-valued function $(Y_1, \dots, Y_N)' = [Y]_N$ can itself be considered as a system of competing particles, with driving function

$$\overline{X} = (X_1, X_2, \dots, X_{N-1}, X_N - q_N^- L_{(N,N+1)}(t))'$$

and parameters of collision $(q_n^{\pm})_{1\le n\le N}$. Since $L_{(N,N+1)}(0) = 0$, and $L_{(N,N+1)}$ is nondecreasing, we have:

$$\overline{X}(0) = X(0),\ \overline{X}(t) - \overline{X}(s) \le X(t) - X(s),\ 0 \le s \le t.$$

Therefore, by Theorem 4.2.2, we get: $[Y(t)]_N \le \overline{Y}(t)$, which proves (4.6). The functions $W$ and $\overline{W}$, defined in (4.3), satisfy

$$\overline{W}(0) = W(0),\ \overline{W}(t) - \overline{W}(s) \le W(t) - W(s),\ 0 \le s \le t.$$

Apply Corollary 4.3.2 to prove (4.4) and (4.5). This completes the proof. □

If we remove particles from both the left and the right, then there are less collisions, so the remaining collision terms decrease and the remaining gaps increase. But we cannot say anything about the remaining particles themselves (whether they shift to the left or to the right). Removal of a few particles from the right eliminates some push from the right; similarly, removal of a few particles from the left eliminates some push from the left. But we cannot say which of these two effects outweighs the other one.

**Corollary 4.3.7.** *Fix $1 \le N_1 < N_2 \le M$. Fix a continuous function $X : \mathbb{R}_+ \to \mathbb{R}^M$ with $X(0) \in \mathcal{W}_M$. Let $Y$ be the system of $N$ competing particles with parameters of collision $(q_n^{\pm})_{1\le n\le M}$ and driving function $X$. Let $\overline{Y} = (\overline{Y}_{N_1}, \ldots, \overline{Y}_{N_2})'$ be the system of $N_2 - N_1 + 1$ competing particles with parameters of collision $(q_n^{\pm})_{N_1\le n\le N_2}$ and driving function $(X_{N_1}, \ldots, X_{N_2})'$. Let $Z = (Z_1, \ldots, Z_{M-1})'$ and $\overline{Z} = (Z_{N_1}, \ldots, Z_{N_2-1})'$ be the corresponding gap processes, and let*

$$L = (L_{(1,2)}, \ldots, L_{(M-1,M)})',\ \overline{L} = (\overline{L}_{(N_1,N_1+1)}, \ldots, \overline{L}_{(N_2-1,N_2)})',$$

*be the vectors of collision terms. Then*

$$Z_k(t) \le \overline{Z}_k(t),\ k = N_1, \ldots, N_2 - 1,\ t \ge 0;$$

$$L_{(k,k+1)}(t) - L_{(k,k+1)}(s) \ge \overline{L}_{(k,k+1)} - \overline{L}_{(k,k+1)}(s),\ k = N_1, \ldots, N_2 - 1,\ 0 \le s \le t.$$

The rest of the corollaries deal with competing Brownian particles. The first of these corollaries is a Brownian counterpart of Corollary 4.3.6. It says that if you remove a few competing Brownian particles from the right, then the remaining particles shift to the right, the local times of collisions decrease, and the gaps increase. This corollary was mentioned in the Introduction, subsection 1.2.

**Corollary 4.3.8.** *Fix $1 < N \le M$. Take a system $Y$ of $M$ competing Brownian particles with parameters $(g_k)_{1\le k\le M}$, $(\sigma_k^2)_{1\le k\le M}$, $(q_k^{\pm})_{1\le k\le M}$, starting from $y \in \mathcal{W}_M$. Let $B_1, \ldots, B_M$ be the corresponding driving Brownian motions. Take another system $\overline{Y}$ of $N$ competing Brownian particles with parameters $(g_k)_{1\le k\le N}$, $(\sigma_k^2)_{1\le k\le N}$, $(q_k^{\pm})_{1\le k\le N}$, starting from $[y]_N$, with driving Brownian motions $B_1, \ldots, B_N$. Let $Z$, $\overline{Z}$ be the corresponding gap processes, and let $L$, $\overline{L}$ be the corresponding vectors of collision local times. Then*

$$Y_k(t) \le \overline{Y}_k(t),\ k = 1, \ldots, N,\ t \ge 0; \tag{4.7}$$

$$Z_k(t) \le \overline{Z}_k(t),\ k = 1, \ldots, N-1,\ t \ge 0; \tag{4.8}$$

$$L_{(k,k+1)}(t) - L_{(k,k+1)}(s) \ge \overline{L}_{(k,k+1)}(t) - \overline{L}_{(k,k+1)}(s),\ k = 1, \ldots, N-1,\ 0 \le s \le t. \tag{4.9}$$

The next corollary is a Brownian counterpart of Corollary 4.3.7. It says that if you remove a few competing Brownian particles from the right and from the left simultaneously, then the local times of collisions decrease, and the gaps increase.

**Corollary 4.3.9.** *Fix $1 \le N_1 < N_2 \le M$. Take a system $Y$ of $M$ competing Brownian particles with parameters $(g_k)_{1\le k\le M}$, $(\sigma_k^2)_{1\le k\le M}$, $(q_k^{\pm})_{1\le k\le M}$, starting from $y \in \mathcal{W}_M$. Let $B_1, \ldots, B_M$ be the corresponding driving Brownian motions. Take another system $\overline{Y} = (\overline{Y}_{N_1}, \ldots, \overline{Y}_{N_2})'$ of $N_2 - N_1 + 1$ competing Brownian particles with parameters $(g_k)_{N_1\le k\le N_2}$, $(\sigma_k^2)_{N_1\le k\le N_2}$, $(q_k^{\pm})_{N_1\le k\le N_2}$, starting from $(y_{N_1}, \ldots, y_{N_2})'$, with driving Brownian motions $B_{N_1}, \ldots, B_{N_2}$. Let $Z = (Z_1, \ldots, Z_{M-1})'$, $\overline{Z} = (\overline{Z}_{N_1}, \ldots, \overline{Z}_{N_2})'$ be the corresponding gap processes, and let $L = (L_{(1,2)}, \ldots, L_{(M-1,M)})'$, $\overline{L} = (L_{(N_1,N_1+1)}, \ldots, L_{(N_2-1,N_2)})'$ be the corresponding vectors of collision terms. Then*

$$Z_k(t) \le \overline{Z}_k(t),\ k = N_1, \ldots, N_2 - 1,\ t \ge 0;$$

$$L_{(k,k+1)}(t) - L_{(k,k+1)}(s) \ge \overline{L}_{(k,k+1)}(t) - \overline{L}_{(k,k+1)}(s),\ k = N_1, \ldots, N_2 - 1,\ 0 \le s \le t.$$

*Remark* 8. We can also remove a few particles from the left instead of the right. We can formulate the statement analogous to Corollary 4.3.6. This fits into the framework of Corollary 4.3.7 when $N_2 = M$. The inequalities (4.4) and (4.5) remain true, and the inequality (4.6) changes sign. Similarly, Corollary 4.3.8 can be modified when we remove particles from the left instead of the right. This fits into the framework of Corollary 4.3.9 when $N_2 = M$.Then the inequality (4.7) changes sign, and the inequalities (4.8) and (4.9) stay true.

*Remark* 9. Corollaries 4.3.6, 4.3.7, 4.3.8 and 4.3.9 can be generalized for the case of infinite particle systems, when $M = \infty$. Recall that we introduced infinite systems of competing particles (and including competing Brownian particles) in Definition 18. Again, here we do not prove existence of these infinite systems; we state these corollaries, assuming these systems exist. The proofs are the same as for finite $M$, with only trivial adjustments.

The following corollary was also mentioned in the Introduction, subsection 1.2.

**Corollary 4.3.10.** *Take two systems, $Y$ and $\overline{Y}$, of $N$ competing Brownian particles with parameters $(g_k)_{1\le k\le N}$, $(\sigma_k^2)_{1\le k\le N}$, $(q_k^{\pm})_{1\le k\le N}$. Suppose these two systems have the same driving Brownian motions. Let $Z$, $\overline{Z}$ be the corresponding gap processes, and let $L$, $\overline{L}$ be the corresponding vectors of collision terms.*

*(i) If $Y(0) \le \overline{Y}(0)$, then $Y(t) \le \overline{Y}(t),\ t \ge 0$.*

*(ii) If $Z(0) \le \overline{Z}(0)$, then $Z(t) \le \overline{Z}(t),\ t \ge 0$, and $L(t) - L(s) \ge \overline{L}(t) - \overline{L}(s),\ 0 \le s \le t$.*

The last two corollaries show how to compare systems of competing Brownian particles in case of the change in drift coefficients or parameters of collision. The first of these corollaries tells that if you increase $q_1^+, \ldots, q_N^+$, the whole system will shift to the right.

**Corollary 4.3.11.** *Consider two systems $Y$ and $\overline{Y}$ of $N$ competing Brownian particles with common drift and diffusion coefficients $(g_k)_{1\le k\le N}$, $(\sigma_k^2)_{1\le k\le N}$, but different parameters of collision $(q_k^{\pm})_{1\le k\le N}$, $(\overline{q}_k^{\pm})_{1\le k\le N}$, such that $\overline{q}_n^+ \ge q_n^+,\ n = 1, \ldots, N$. Suppose $Y(0) = \overline{Y}(0)$ and the driving Brownian motions are the same for these two systems. Then*

$$Y(t) \le \overline{Y}(t),\ t \ge 0.$$

*Proof.* Let $B_1, \dots, B_N$ be the driving Brownian motions for these systems. Then $Y$ and $\overline{Y}$ are systems of competing particles with parameters of collision $(q_n^{\pm})_{1 \le n \le N}$, $(\overline{q}_n^{\pm})_{1 \le n \le N}$, and the same driving function

$$X(t) = \left(Y_1(0) + g_1 t + \sigma_1 B_1(t), \dots, Y_N(0) + g_N t + \sigma_N B_N(t)\right)'.$$

Apply Theorem 4.2.2 and finish the proof. □

The following corollary shows how to use the drift coefficients for comparison.

**Corollary 4.3.12.** *Consider two systems $Y$ and $\overline{Y}$ of $N$ competing Brownian particles with common diffusion coefficients $(\sigma_k^2)_{1 \le k \le N}$ and parameters of collision $(q_n^{\pm})_{1 \le n\ leN}$, but with different drift coefficients $(g_n)_{1 \le n \le N}$, $(\overline{g}_n)_{1 \le n \le N}$. Suppose $Y(0) = \overline{Y}(0)$ and the driving Brownian motions are the same for these two systems. Let $Z$ and $\overline{Z}$ be the corresponding gap processes, and let $L$ and $\overline{L}$ be the corresponding vectors of collision terms.*

*(i) If $g_k \le \overline{g}_k,\ k = 1, \dots, N$, then $Y(t) \le \overline{Y}(t),\ t \ge 0$.*

*(ii) If $g_{k+1} - g_k \le \overline{g}_{k+1} - \overline{g}_k,\ k = 1, \dots, N-1$, then*

$$Z(t) \le \overline{Z}(t),\ t \ge 0; \quad L(t) - L(s) \ge \overline{L}(t) - \overline{L}(s),\ 0 \le s \le t.$$

*Proof.* Let $B_1, \dots, B_N$ be the driving Brownian motions for these systems. Then $Y$ and $\overline{Y}$ are systems of competing particles with parameters of collision $(q_n^{\pm})_{1 \le n \le N}$ and driving functions

$$X(t) = \left(Y_1(0) + g_1 t + \sigma_1 B_1(t), \dots, Y_N(0) + g_N t + \sigma_N B_N(t)\right)',$$

$$\overline{X}(t) = \left(Y_1(0) + \overline{g}_1 t + \sigma_1 B_1(t), \dots, Y_N(0) + \overline{g}_N t + \sigma_N B_N(t)\right)'.$$

(i) We have: $X(t) - X(s) \le \overline{X}(t) - \overline{X}(s),\ 0 \le s \le t$, and $X(0) = \overline{X}(0)$. Apply Theorem 4.2.2 and finish the proof.

(ii) This statement immediately follows from Corollary 4.3.2. □

In each of the last few corollaries, if we remove the requirement that the driving Brownian motions must be the same, then we get stochastic comparison instead of pathwise comparison.

## 4.4 Proofs of Theorems 4.2.1 and 4.2.2

### 4.4.1 Outline of the proofs

We prove Theorems 4.2.1 and 4.2.2 by approximating the general continuous driving functions by "simple" functions, which are defined as follows.

**Definition 21.** A continuous function $f : [0,T] \to \mathbb{R}^d$ is called *regular* if it is piecewise linear with each piece parallel to one of the coordinate axes; that is, if there exist a partition $0 = t_0 < t_1 < \ldots < t_N = T$ and numbers $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$, $j_1, \ldots, j_N \in \{1, \ldots, d\}$ such that for $k = 1, \ldots, N$, we have:

$$f(t) = f(t_k) + \alpha_k e_{j_k}(t - t_{k-1}),\ t_{k-1} \le t \le t_k.$$

Two regular functions $f$ and $\overline{f}$ are called *coupled* if the partition $t_0, \ldots, t_N$ and the indices $j_1, \ldots, j_N$ are the same for them.

We make three observations:

(i) Any continuous function $X : [0,T] \to \mathbb{R}^d$ can be uniformly approximated by regular functions. This is proved in Lemma 4.4.1. Moreover, we show that a pair of continuous functions $X$ and $\overline{X}$ which satisfy (4.1) can be uniformly on $[0,T]$ approximated by a pair of coupled regular functions so that within each pair two regular functions also satisfy (4.1). This is proved in Lemma 4.4.2.

(ii) All the objects we are considering in this chapter (the solution to the Skorohod problem in the orthant, boundary terms in the Skorohod problem, the system of competing particles, the gap process, the vector of collision terms) continuously depend on the corresponding driving functions; see Lemma 4.4.3 and Lemma 4.4.4. In fact, that this is true for the Skorohod problem (both for the solution and for the boundary terms) was already shown in [125], [51]; we just restate it here. So we can prove Theorems 4.2.1 and 4.2.2 just for regular driving functions.

(iii) In Lemmas 4.4.6 and 4.4.7, we show that solutions to the Skorohod problem and systems of competing particles are "memoryless": if you take a moment $t > 0$, then their behavior after this moment depends only on their current position and future dynamics of the driving function. This is very similar to Markov property (although the concepts of the Skorohod problem and competing particles are deterministic, not random). This allows us to consider driving regular functions (and the solutions) piece by piece.

The goal of these three observations is Lemma 4.4.8. It shows that Theorems 4.2.1 and 4.2.2 can be reduced to the case when the driving functions are not just piecewise linear, but exactly linear, with the directional vector parallel to one of the axes. And since they are coupled, this axis is the same for both functions. That is, we can consider

$$X(t) = x + \alpha e_i t,\ \overline{X}(t) = \overline{x} + \overline{\alpha} e_i t, \tag{4.10}$$

where $\alpha, \overline{\alpha} \in \mathbb{R}$, $i = 1, \ldots, d$. The condition (4.1) for these functions is equivalent to

$$x \le \overline{x}, \quad \alpha \le \overline{\alpha}. \tag{4.11}$$

But for regular linear driving functions as in (4.10), we can actually solve the Skorohod problem explicitly, and find the solution $Z$ and and the vector of boundary terms $L$ in exact form. This is done in Lemma 4.4.9. We can do the same for the system $Y$ of competing particles: Lemma 4.4.11. Then we can manually compare the solutions $Z$ and $\overline{Z}$ of the Skorohod problem, and the vectors $L$ and $\overline{L}$ of boundary terms, or (if we are considering systems of competing particles) $Y$ and $\overline{Y}$. This completes the proof of Theorems 4.2.1 and 4.2.2.

The rest of this section is organized as follows.

In subsection 4.2, we state and prove the technical results mentioned above: (i) approximation of continuous driving functions by regular functions; (ii) continuous dependence on driving functions; (iii) the memoryless property. In subsection 4.3, we explicitly solve the

Skorohod problem for regular driving functions in Lemma 4.4.9 and find the solution together with the boundary terms. In subsection 4.4, we do the same for a system of competing particles in Lemma 4.4.11. In subsections 4.5 and 4.6, we prove Theorems 4.2.1 and 4.2.2 for regular linear driving functions. This completes the proof.

### *4.4.2 Auxillary results*

*Observation (i): approximation by regular driving functions.*

**Lemma 4.4.1.** *Fix $T \ge 0$ and take a continuous function $X : [0,T] \to \mathbb{R}^d$. Then there exists a sequence $(X^{(n)})_{n\ge1}$ of regular functions $[0,T] \to \mathbb{R}^d$ which uniformly converges to $X$ on $[0,T]$.*

*Proof.* Let

$$t_i := \frac{Ti}{d},\ i = 0, \dots, d.$$

Split the interval $[0,T]$ into $d$ equal subintervals: $I_i := [t_{i-1}, t_i]$, $i = 1, \dots, d$. On the $i$th subinterval $I_i$, define the function $X^{(1)}$ as follows:

$$X^{(1)}(t) = X^{(1)}\left(t_{i-1}\right) + \left(X_i\left(t_i\right) - X_i\left(t_{i-1}\right)\right)\frac{t - t_{i-1}}{t_i - t_{i-1}}e_i,\ i = 1, \dots, d,\quad a \le t \le b.$$

Then $X^{(1)}(0) = X(0)$ and $X^{(1)}(T) = X(T)$. During the time interval $I_i$, only the $i$th component of the function $X^{(1)}$ is changing; other components stay constant. The $i$th component $X_i^{(1)}$ is moving between $X_1(0)$ and $X_1(T)$. So

$$\left|X_i^{(1)}(t) - X_i(0)\right| \le \left|X_i(T) - X_i(0)\right|,\quad t \in [0,T].$$

Therefore,

$$\|X^{(1)}(t) - X(0)\| \le \|X(T) - X(0)\|,\quad t \in [0,T],$$

and

$$\|X^{(1)}(t) - X(t)\| \le \|X(T) - X(0)\| + \max_{0\le t\le T}\|X(t) - X(0)\| \le 2\max_{0\le t\le T}\|X(t) - X(0)\|.$$

Let $s_k := kT/n,\ k = 0, \ldots, n$. Split $[0, T]$ into $n$ equal subintervals $J_k = [s_{k-1}, s_k],\ k = 1, \ldots, n$, and perform the same construction of $X^{(1)}$ for each of these small subintervals in place of $[0, T]$. Then we get a continuous function $X^{(n)}$ such that

$$X^{(n)}(s_k) = X(s_k), \quad k = 0, \ldots, n.$$

For $t \in J_k$, we have:

$$\|X^{(1)}(t) - X(t)\| \le 2 \max_{s_{k-1} \le t \le s_k} \|X(t) - X\left(s_{k-1}\right)\|.$$

Therefore,

$$\max_{0 \le t \le T} \|X^{(n)}(t) - X(t)\| \le 2 \max_{k=1,\ldots,n} \max_{s_{k-1} \le t \le s_k} \|X(kT/n) - X((k-1)T/n)\|. \tag{4.12}$$

But the function $X$ is uniformly continuous on $[0, T]$. Therefore, the right-hand side of (4.12) tends to zero as $n \to \infty$. Thus, the sequence of regular functions $(X^{(n)})_{n \ge 1}$ uniformly converges to $X$. $\square$

We will call the sequence constructed in Lemma 4.4.1 *the standard approximating sequence.*

**Lemma 4.4.2.** *Fix $T \ge 0$ and take two continuous functions $X, \overline{X} : [0, T] \to \mathbb{R}^d$ such that*

$$X(0) \le \overline{X}(0); \quad X(t) - X(s) \le \overline{X}(t) - \overline{X}(s), \quad 0 \le s \le t \le T.$$

*Then there exist two sequences $(X^{(n)})_{n \ge 1}$, $(\overline{X}^{(n)})_{n \ge 1}$ of regular functions $[0, T] \to \mathbb{R}^d$ such that:*

*(i) $X^{(n)} \to X$, $\overline{X}^{(n)} \to \overline{X}$ uniformly on $[0, T]$ as $n \to \infty$;*

*(ii) for every $n \ge 1$, the functions $X^{(n)}$ and $\overline{X}^{(n)}$ are coupled;*

*(iii) $X^{(n)}(0) \le \overline{X}^{(n)}(0)$ and $X^{(n)}(t) - X^{(n)}(s) \le \overline{X}^{(n)}(t) - \overline{X}^{(n)}(s)$ for all $0 \le s \le t \le T$.*

*Proof.* Construct two standard approximating sequences as in the proof of Lemma 4.4.1. Let us show that

$$X^{(1)}(t) - X^{(1)}(s) \le \overline{X}^{(1)}(t) - \overline{X}^{(1)}(s),\ 0 \le s \le t.$$

Indeed, $X^{(1)}$ and $\overline{X}^{(1)}$ are linear on each $[(k-1)T/N, kT/N]$, and

$$X^{(1)}\left(\frac{kT}{N}\right) - X^{(1)}\left(\frac{(k-1)T}{N}\right) \le \overline{X}^{(1)}\left(\frac{kT}{N}\right) - \overline{X}^{(1)}\left(\frac{(k-1)T}{N}\right).$$

The proof is similar for $X^{(n)}$ and $\overline{X}^{(n)}$ instead of $X^{(1)}$ and $\overline{X}^{(1)}$. $\square$

*Observation (ii): continuous dependence.* The first result, about the Skorohod problem in the orthant, is already known from [51], [127]; see also [125].

**Lemma 4.4.3.** *Fix $d \ge 1$, the dimension, and let $S = \mathbb{R}^d_+$. Take a $d \times d$ reflection nonsingular $\mathcal{M}$-matrix $R$. Consider a continuous function $X : \mathbb{R}_+ \to \mathbb{R}^d$ with $X(0) \in S$, and let $Z$ be the solution of the Skorohod problem in $S$ with reflection matrix $R$ and driving function $X$. Let $L$ be the vector of boundary terms. The mapping $X \mapsto (Z, L)$ is continuous in the topology of uniform convergence on $[0, T]$, for every $T > 0$.*

A counterpart of the previous theorem is the continuity result about systems of competing particles.

**Lemma 4.4.4.** *Fix $N \ge 2$. Consider the parameters of collision $(q_k^{\pm})_{1 \le k \le N}$. Consider a continuous function $X : \mathbb{R}_+ \to \mathbb{R}^N$ with $X(0) \in \mathcal{W}_N$, and let $Y$ be the system of $N$ competing particles with driving function $X$ and the given parameters of collision. Let $L$ be the vector of collision terms. Then the mapping $X \mapsto (Y, L)$ is continuous in the topology of uniform convergence on $[0, T]$, for every $T > 0$.*

*Proof.* Return to the proof of Lemma 3.4.3. The mapping

$$(X_1, \dots, X_N)' \mapsto (X_2 - X_1, \dots, X_N - X_{N-1})'$$

is continuous in this topology. The mapping

$$(X_2 - X_1, \dots, X_N - X_{N-1})' \mapsto (Z_1, \dots, Z_{N-1})'$$

is continuous, by Lemma 4.4.4 just above. The mappings $(X_1, \dots, X_N)' \mapsto \alpha_1 X_1(t) + \ldots + \alpha_N X_N(t)$ and $\mathcal{Y}(t) \mapsto C^{-1}\mathcal{Y}(t)$ are continuous. The composition of all these continuous mappings is the mapping $X \mapsto Y$, which is also continuous. Similarly, $X \mapsto L$ is continuous. $\square$

These continuity results, together with the approximation results (Lemma 4.4.1 and Corollary 4.4.2), allow us to substantially narrow the class of driving functions. Let us state this as a separate lemma.

**Lemma 4.4.5.** *If Theorems 4.2.1 and 4.2.2 are true for coupled regular driving functions, then they are true in the general case.*

*Observation (iii): memoryless property.* This allows us to further narrow the scope of driving functions: to take coupled regular *linear* driving functions.

**Lemma 4.4.6.** *Fix $d \ge 1$. Take a continuous function $X : \mathbb{R}_+ \to \mathbb{R}^d$ with $X(0) \in S = \mathbb{R}^d_+$ and a $d \times d$-reflection nonsingular matrix $R$. Let $Z$ be the solution of the Skorohod problem in $S$ with reflection matrix $R$ and driving function $X$. Let $L$ be the vector of boundary terms. Fix $T \ge 0$. For $t \ge 0$, let*

$$X_T(t) = X(T+t) - X(T) + Z(T),$$

$$L_T(t) = L(T+t) - L(T),\ Z_T(t) = Z(T+t).$$

*Then $Z_T$ is the solution of the Skorohod problem with reflection matrix $R$ and driving function $X_T$, and $L_T$ is the corresponding vector of boundary terms.*

*Proof.* It suffices to check the definition: we need to prove that

$$Z_T(t) = X_T(t) + RL_T(t),\ t \ge 0.$$

This follows from

$$Z(t+T) = X(t+T) + RL(t+T) \text{ and } Z(T) = X(T) + RL(T).$$

$\square$

Let us state a similar property for systems of competing particles. The proof is similar to the previous one and is omitted.

**Lemma 4.4.7.** *Fix $N \ge 2$. Assume $Y$ is a system of $N$ competing particles with driving function $X : \mathbb{R}_+ \to \mathbb{R}^N$ and parameters of collision $(q_n^{\pm})_{1\le n\le N}$. Let $L$ be the corresponding vector of collision terms. Fix $T \ge 0$. For $t \ge 0$, let*

$$X_T(t) = X(T+t) - X(T) + Y(T),$$

$$L_T(t) = L(T+t) - L(T),\ Y_T(t) = Y(T+t).$$

*Then the function $Y_T$ is a system of $N$ competing particles with driving function $X_T$ and the parameters of collision, and $L_T$ is the corresponding vector of collision terms.*

*Remark* 10. The memoryless property also holds true for infinite systems of competing particles from Definition 18. The proof is the same, with obvious adjustments.

The memoryless property allows us to narrow the class of driving functions to *regular linear* functions, that is, functions of the type (4.10).

**Lemma 4.4.8.** *If Theorems* 4.2.1 *and* 4.2.2 *are true for coupled regular linear driving functions as in* (4.10)*, satisfying* (4.11)*, they are true in the general case.*

*Proof.* By Lemma 4.4.5, it suffices to show these theorems for coupled regular driving functions. For example, let us prove Theorem 4.2.1 for coupled regular driving functions $X$ and $\overline{X}$; Theorem 4.2.2 is proved similarly. Let $0 = t_0 < t_1 < \ldots < t_N = T$ and $j_1, \ldots, j_N$ be the common parameters for these functions, as in Definition 21. The restrictions

$$X|_{[t_0,t_1]},\ \overline{X}|_{[t_0,t_1]}$$

are coupled regular linear functions. Assuming we proved Theorem 4.2.1 for such driving functions, we have:

$$Z(t) \le \overline{Z}(t),\ t \ge 0;\ L(t) - L(s) \ge \overline{L}(t) - \overline{L}(s),\ 0 \le s \le t \le t_1.$$

In particular, we have: $Z(t_1) \le \overline{Z}(t_1)$. But $t \mapsto Z(t+t_1)$ is the solution of the Skorohod problem with reflection matrix $R$ and driving function $t \mapsto X(t+t_1) - X(t_1) + Z(t_1)$; a similar statement is true for $t \mapsto \overline{Z}(t+t_1)$. And

$$L(t+t_1) - L(t_1),\ \overline{L}(t+t_1) - \overline{L}(t_1),\ 0 \le t \le t_2 - t_1.$$

are the corresponding vectors boundary terms for these Skorohod problems. The functions

$$X(t+t_1) - X(t_1) + Z(t_1) \ \text{ and } \ \overline{X}(t+t_1) - \overline{X}(t_1) + \overline{Z}(t_1) \tag{4.13}$$

are coupled regular linear driving functions on $[0, t_2 - t_1]$. They also satisfy conditions of Theorem 4.2.1. Indeed,

$$X(t+t_1) - X(t_1) + Z(t_1)\big|_{t=0} = Z(t_1) \in S,\ \overline{X}(t+t_1) - \overline{X}(t_1) + \overline{Z}(t_1)\big|_{t=0} = \overline{Z}(t_1) \in S,$$

and for $0 \le s \le t \le t_2 - t_1$ we have:

$$\begin{aligned}(X(t+t_1) - X(t_1) + Z(t_1)) - (X(s+t_1) - X(t_1) + Z(t_1)) = X(t+t_1) - X(s+t_1)\\ \le \overline{X}(t+t_1) - \overline{X}(s+t_1) = (\overline{X}(t+t_1) - \overline{X}(t_1) + \overline{Z}(t_1)) - (\overline{X}(s+t_1) - \overline{X}(t_1) + \overline{Z}(t_1)).\end{aligned}$$

Therefore, applying Theorem 4.2.1 for coupled regular linear driving functions (4.13), we get:

$$Z(t+t_1) \le \overline{Z}(t+t_1),\ 0 \le t \le t_2 - t_1,$$

$$L(t+t_1) - L(s+t_1) \ge \overline{L}(t+t_1) - \overline{L}(s+t_1),\ 0 \le s \le t \le t_2 - t_1.$$

Similarly, moving to the next interval $[t_2, t_3]$, etc., we can show that for every $k = 1, \dots, N$,

$$Z(t) \le \overline{Z}(t),\ t \in [t_{k-1}, t_k], \tag{4.14}$$

$$L(t+t_{k-1}) - L(s+t_{k-1}) \ge \overline{L}(t+t_{k-1}) - \overline{L}(s+t_{k-1}),\ 0 \le s \le t \le t_k - t_{k-1}. \tag{4.15}$$

We can equivalently write (4.14) as

$$Z(t) \le \overline{Z}(t),\ t \in [0, T],$$

and (4.15) as

$$L(t) - L(s) \geq \overline{L}(t) - \overline{L}(s),\ t_{k-1} \leq s \leq t \leq t_k,\ k = 1, \ldots, N. \tag{4.16}$$

Now, let us show that

$$L(t) - L(s) \geq \overline{L}(t) - \overline{L}(s),\ 0 \leq s \leq t \leq T.$$

This is done just by summing the inequalities (4.16): find $k, l = 1, \ldots, N$ such that

$$t_{k-1} \leq s \leq t_k \leq \ldots \leq t_l \leq t \leq t_{l+1}.$$

Then we have:

$$\begin{cases} L(t) - L(t_l) \geq \overline{L}(t) - \overline{L}(t_l) \\ L(t_l) - L(t_{l-1}) \geq \overline{L}(t_l) - \overline{L}(t_{l-1}) \\ \ldots \\ L(s) - L(t_{k-1}) \geq \overline{L}(s) - \overline{L}(t_{k-1}) \end{cases}$$

Sum these inequalities and finish the proof. □

### *4.4.3 Exact solutions of the Skorohod problem for regular linear driving functions*

Fix the dimension $d \geq 1$, and recall that $S = \mathbb{R}^d_+$ is the positive $d$-dimensional orthant. Let

$$X(t) = x + \alpha e_i t,\ 0 \leq t \leq T, \tag{4.17}$$

be a regular linear driving function. Here, $x \in S$, $\alpha \in \mathbb{R}$ and $i = 1, \ldots, d$. Take a reflection nonsingular $\mathcal{M}$-matrix $R$. In this subsection, we find the explicit solution $Z$ (and the vector $L$ of boundary terms) for the Skorohod problem in the orthant $S$ with reflection matrix $R$ and driving function $X$.

Let us first describe the behavior of this solution informally. The solution $Z$ "wants" to move along with the driving function $X$. However, if $X$ gets out of the orthant $S$, then $Z$ "is not allowed" out of the orthant; the boundary terms push it back to $S$.

*Case 1.* $\alpha \geq 0$. Then $X$ does not get out of $S$. This is a trivial case: the boundary terms $L_i$ stay zero: $L(t) \equiv 0$, and the solution $Z$ exactly follows the driving function $X$: $Z(t) \equiv X(t)$.

*Case 2.* $\alpha < 0$ and $x_i = 0$. Then the driving function $X$ is moving along the axis $x_i$ in the negative direction, starting from the face $S_i$ of the boundary $\partial S$. The solution $Z$ of the Skorohod problem "wants" to move in tandem with $X$, which means that it "wants" to cross this face $S_i$. However, it cannot do this, since it must be in the orthant. Therefore, it stays at this face. The boundary term $L_i$ increases: this term "counters the influence" of the driving function $X$, which "wants" to take $Z$ out of the orthant. This increase in $L_i$ also influences other components $Z_j,\ j \neq i$, of $Z_i$, through reflection matrix $R$ (or, more precisely, through the elements $r_{ij} \leq 0,\ j \neq i$). Therefore, if $Z$ moves on the face $S_i$, this contributes to *decrease* of other components $Z_j,\ j \neq i$. Let

$$I(t) = \{j = 1, \dots, d \mid Z_j(t) = 0\}. \tag{4.18}$$

Suppose $j \in I(0)$. Then $Z_j$ was originally zero, and it "wants" to decrease because of the increase in $L_i$. But $Z_j$ cannot decrease further, because $Z(t)$ must stay in the orthant. Therefore, the boundary term $L_j$ starts to increase, to "counter" the influence of $L_i$. This can be said of all $j \in I(0)$. If, however, $j \notin I(0)$, then $Z_j(0) > 0$, and so $Z_j$ "is allowed" to decrease, so the boundary term $L_j$ stays zero.

Let us summarize this: for $j \in I(t)$, the boundary term $L_j$ increases, and $Z_j(t) = 0$; for $j \notin I(t)$, the boundary term $L_j(t) = 0$, and $Z_j$ decreases. This description is accurate until some new component $Z_j$ hits zero; another way to say this is when the set-valued function $I$ jumps upward. Denote this moment by $\tau_1$. Then, using the memoryless property from Lemma 4.4.6, we repeat the same, starting from $\tau_1$. Let $\tau_2$ be the next jump moment of the function $I$, etc. Between any of these two moments, the function $I$ is constant. There will be no more than $d$ pf these moments, because the function $I$ increases at every jump, and $i \in I(0)$, but $I(t) \subseteq \{1, \dots, d\}$.

*Case 3.* $x_i > 0$ and $\alpha < 0$. Then $X$ moves to the boundary and hits it at some moment

$\tau_1 = x_1/|\alpha|$. The solution $Z$ "wants" to move in tandem with $Z$. Until $\tau_1$, however, the solution $Z$ does not need to be pushed inside the orthant $S$ by boundary terms, so this is also a trivial case: $L(t) \equiv 0$, $Z(t) \equiv X(t)$. If $\tau_1 \ge T$, then the time-horizon is earlier than hitting moment of the boundary, and this completes the description of $Z$ and $L$. If $\tau_1 < T$, then we use the memoryless property and start from $\tau_1$; we are back in Case 2.

Now, let us formulate the result rigorously.

**Lemma 4.4.9.** *Let $R$ be a $d \times d$ reflection nonsingular $\mathcal{M}$-matrix. Let $X$ be given by* (4.17)*. Let $Z$ be the solution to the Skorohod problem in the orthant $S$ with reflection matrix $R$ and driving function $X$. Let $L$ be the corresponding vector of boundary terms. Then $Z$ and $L$ are given by the following formulas.*

*(i) If $\alpha \ge 0$, then $Z(t) \equiv X(t)$ and $L(t) \equiv 0$;*

*(ii) If $\alpha < 0$, and $x_i = 0$, then:*

*(a) $Z$ is nondecreasing, $L$ is nondecreasing, the set-valued function $I$ defined in* (4.18) *is nondecreasing;*

*(b) there exists a sequence $0 = \tau_0 < \tau_1 < \ldots < \tau_K = T$ of moments such that on each $[\tau_{l-1}, \tau_l)$, $I(t)$ is constant, and*

$$\tau_l := \inf\{t > \tau_{l-1} \mid I(t) \ne I(\tau_{l-1})\} \wedge T. \tag{4.19}$$

*We use the convention* $\inf \varnothing = +\infty$*. At each moment $\tau_l$, $l = 1, \ldots, K-1$, the function $I$ jumps and increases.*

*(c) For $t \in [\tau_{l-1}, \tau_l]$, letting $J := I(\tau_{l-1})$, we have:*

$$[Z(t)]_J = 0; \quad [Z(t)]_{J^c} = [Z(\tau_{l-1})]_{J^c} + |\alpha| [R]_{J^c J} [R]_J^{-1} [e_i]_J (t - \tau_{l-1}), \tag{4.20}$$

$$[L(t)]_J = [L(\tau_{l-1})]_J + |\alpha| [R]_J^{-1} (t - \tau_{l-1}); \quad [L(t)]_{J^c} = [L(\tau_{l-1})]_{J^c}. \tag{4.21}$$

*(iii) If $\alpha < 0$, and $x_i > 0$, then $Z$ is nondecreasing, $L$ is nondecreasing, the set-valued function $I$ from* (4.18) *is nondecreasing, and there exists a sequence $0 = \tau_0 < \tau_1 < \ldots <$*

$\tau_K = T$ *of moments such that on each* $[\tau_{l-1}, \tau_l)$, $I(t) \equiv I(\tau_{l-1}) =: J$ *is constant, on* $[0, \tau_1]$ *we have:*

$$Z(t) \equiv X(t), \quad L(t) \equiv 0,$$

*and on* $[\tau_{l-1}, \tau_l]$, $l = 2, \ldots, k$, *the functions* $Z$ *and* $L$ *are given by the formulas* (4.20) *and* (4.21). *The equation* (4.19) *is still true. As in case (ii), at each moment* $\tau_l$, $l = 1, \ldots, K-1$, *the function* $I$ *jumps and increases.*

*Proof.* The case (i) is straightforward. Let us show (ii). Using the memoryless property and induction by $l$, we can assume w.l.o.g. that $\tau_l = 0$, that is, $l = 0$: it suffices to consider only the first interval $[0, \tau_1]$ of linearity. We have: $x_i = 0$, that is, $i \in I(0) = J$. We can write the main equation governing $Z$ and $L$,

$$Z(t) = X(t) + RL(t),$$

in the block form:

$$\begin{cases} [Z(t)]_J = [X(t)]_J + [R]_J[L(t)]_J + [R]_{JJ^c}[L(t)]_{J^c} \\ [Z(t)]_{J^c} = [X(t)]_{J^c} + [R]_{J^cJ}[L(t)]_J + [R]_{J^c}[L(t)]_{J^c} \end{cases} \tag{4.22}$$

But $[X(t)]_{J^c} = [x + \alpha e_i t]_{J^c} = [x]_{J^c}$, because $i \in J$ and $i \notin J^c$. Also, $[X(t)]_J = \alpha[e_i]_J t$, because $x_i = 0$. Now it is straightforward to check that the functions $Z(t)$ and $L(t)$ given by

$$[Z(t)]_J = 0, \quad [Z(t)]_{J^c} = |\alpha|[R]_{J^cJ}[R]_J^{-1}t + [x]_{J^c},$$

$$[L(t)]_{J^c} = 0, \quad [L(t)]_J = |\alpha|[R]_J^{-1}[e_i]_J t,$$

satisfy the system (4.22). Let us now verify that for $j = 1, \ldots, d$, the boundary term $L_j$ can grow only when $Z_j = 0$. This follows from the fact that

$$Z_j(t) \equiv 0, \ j \in J; \ L_j(t) \equiv 0, \ j \in J^c.$$

The next step is to check that $L$ is nondecreasing and $Z$ is nonincreasing on $[\tau_0, \tau_1]$. Indeed, by Lemma 4.6.1 $[R]_J^{-1} \geq 0$, and $[e_i]_J \geq 0$, so

$$|\alpha|[R]_J^{-1}[e_i]_J \geq 0. \tag{4.23}$$

Therefore, $L$ is nondecreasing on $[0, \tau_1]$. Next, $R$ is a reflection nonsingular $\mathcal{M}$-matrix, so off-diagonal elements of $R$ (in particular, all elements of $[R]_{J^cJ}$) are nonpositive. From this and (4.23) it follows that

$$|\alpha|[R]_{J^cJ}[R]_J^{-1}[e_i]_J \le 0.$$

So $Z$ is nonincreasing on $[0, \tau_1]$. We have the formula

$$\tau_1 := \inf\{t \ge 0 \mid I(t) \ne I(0)\} \wedge T,$$

so $\tau_1$ is the first moment (no later than the time horizon $T$) when $Z$ "new" parts of the boundary, and the function $I$ increases. If this moment comes later than $T$, then we let $\tau_1 = T$. By definition of $\tau_1$, we have: $I(0) \subsetneq I(\tau_1)$. So the set-valued function $I$ is constant on $[0, \tau_1)$, but increases by a jump at $\tau_1$.

Part (iii) follows from (ii) and the memoryless property. □

### *4.4.4 Exact formulas for a system of competing particles with a regular linear driving function*

Let us now do a similar calculation as in the previous subsection, but for a system of competing particles instead of a Skorohod problem. First, let us informally describe the dynamics of these particles. Recall that the driving function is given by (4.17).

Without loss of generality, assume $\alpha > 0$. The case $\alpha = 0$ is trivial ($Y(t) \equiv X(t) \equiv x$ and $L(t) \equiv 0$), and the case $\alpha < 0$ can be reduced to $\alpha > 0$ by the following lemma. (The proof is trivial and is omitted.)

**Lemma 4.4.10.** *Suppose $Y = (Y(t), t \ge 0)$ is a system of $N$ competing particles with parameters of collision $(q_k^{\pm})_{1 \le k \le N}$ and driving function $X$. Then $-Y := (-Y(t), t \ge 0)$ is also a system of $N$ competing particles with parameters of collision $(\tilde{q}_n^{\pm})_{1 \le n \le N}$, where*

$$\tilde{q}_n^+ = q_{N-n+1}^-,\ \tilde{q}_n^- = q_{N-n+1}^+, \quad n = 1, \dots, N,$$

*and driving function $-X$.*

A system of competing particles involves colliding particles, and "asymmetric collisions" means that they "have different mass". We can rewrite the expression

$$X(t) = x + \alpha e_i t$$

in the coordinate form:

$$X_i(t) = x_i + \alpha t,\ X_j(t) = x_j,\ j \ne i.$$

This means that the $i$th ranked particle "wants" to move to the right with speed $\alpha$, and all other particles "want" to stay motionless. But when the particles, say with ranks $i$ and $i+1$, collide, they move together to the right with a new speed (smaller than $\alpha$). The collision term for particles $Y_i$ and $Y_{i+1}$ starts to increase linearly from zero when they first collide. All other particles stay motionless. When these two particles hit, say, the $i+2$nd particle $Y_{i+2}$, then these three particles stick together and move to the right. The collision terms $L_{(j,j+1)}$ for all other pairs of adjacent particles $Y_j, Y_{j+1}$ stay zero. Indeed, even if $Y_j(t) = Y_{j+1}(t)$, but $Y_j$ and $Y_{j+1}$ are not moving, then no collision term is required to keep them in order: $Y_j(t) \le Y_{j+1}(t)$. But the collision term $L_{(i+1,i+2)}$ starts to increase, and the collision term $L_{(i,i+1)}$ continues to increase.

In other words, at any time $t$ there is a set

$$I(t) = \{j = i, \dots, N \mid Y_j(t) = Y_i(t)\} \tag{4.24}$$

of particles which are moving together with $Y_i$ to the right at this moment $t$. Since these particles satisfy

$$Y_1(t) \le Y_2(t) \le \dots \le Y_N(t),$$

the set $I(t)$ has the form

$$I(t) = \{i, i+1, \dots, k(t)\}$$

for some $k(t) = i, \dots, N$. The speed of this movement depends on $k(t)$. When these moving particles hit a new particle $Y_{k(t)+1}$, then the set $I$ increases by a jump. So we have a sequence of moments of hits:

$$0 = \tau_0 < \tau_1 < \dots < \tau_K = T.$$

At any interval between these moments, $I(t)$ is constant, the particles $Y_j,\ j \in I(t)$ move to the right, and all other particles do not move.

Now, let us do the precise calculation.

**Lemma 4.4.11.** *There exists a sequence of moments*

$$0 = \tau_0 < \tau_1 < \ldots < \tau_K := T$$

*such that on each* $[\tau_{l-1}, \tau_l)$*, the set-valued function* $I$ *defined in* (4.24) *is constant, but it jumps and increases at each* $\tau_l$ *(except maybe* $\tau_K = T$*). On each* $[\tau_{l-1}, \tau_l)$*, define*

$$\beta_l = \alpha\left[1 + \frac{q_i^-}{q_{i+1}^+} + \frac{q_i^- q_{i+1}^-}{q_{i+1}^+ q_{i+2}^+} + \ldots + \frac{q_i^- q_{i+1}^- \ldots q_{k_l-1}^-}{q_{i+1}^+ q_{i+2}^+ \ldots q_{k_l}^+}\right],$$

$k_l \equiv k(t)$ *for* $t \in [\tau_{l-1}, \tau_l)$*. Then we have:*

$$Y_j(t) = \mathrm{const},\ j \in I^c(t);\quad \textit{and}\ \ Y_j(t) \equiv Y_i(t) = Y_i(\tau_{l-1}) + \beta_l(t - \tau_{l-1}),\ j \in I(t). \tag{4.25}$$

*The moment* $\tau_l$ *is defined as*

$$\tau_l = \inf\{t \ge \tau_{l-1} \mid I(t) \ne I(\tau_{l-1})\} \wedge T.$$

*As before, we use the convention* $\inf\varnothing = +\infty$*.*

*Proof.* Similarly to the previous subsection, we can use the memoryless property and induction by $l$ to assume that $l = 0$. Assume $I(0) = \{i, \ldots, k_0\}$, so initially the "leading" particle $i$ was at the same position as particles with ranks $i+1, \ldots, k_0$. Note that we care only about particles with ranks larger than $i$, because the particle with rank $i$ is moving to the right. Even if, say, the particle with rank $i-1$ occupied the same position initially as the particle with rank $i$, they will not interact: the particle $Y_i$, together with $Y_{i+1}, \ldots, Y_{k_0}$, will move rightward and"leave" the idle particle $i-1$ at its place. So we have: on $[0, \tau_1]$,

$$L_{(1,2)}(t) = \ldots = L_{(i-1,i)}(t) = L_{(k_0,k_0+1)}(t) = \ldots = L_{(N-1,N)}(t) = 0,$$

and $Y_1, \dots, Y_{i-1}, Y_{k_0+1}, \dots, Y_N$ are constant on this time interval. The dynamics of the particles $Y_i, \dots, Y_{k_0}$ on $[0, \tau_1]$ is described as follows:

$$\begin{cases} Y_i(t) = Y_{i+1}(t) = \ldots = Y_{k_0}(t), \\ Y_i(t) = x_i + \alpha t - q_i^- L_{(i,i+1)}(t), \\ Y_{i+1}(t) = x_{i+1} + q_{i+1}^+ L_{(i,i+1)}(t) - q_{i+1}^- L_{(i+1,i+2)}(t), \\ \ldots \\ Y_{k_0}(t) = x_{k_0} + q_{k_0}^+ L_{(k_0-1,k_0)}(t). \end{cases}$$

But $x_i = x_{i+1} = \ldots = x_{k_0}$, because $Y_i(0) = Y_{i+1}(0) = \ldots = Y_{k_0}(0)$ (the initial positions of particles with ranks $i, i+1, \dots, k_0$ are the same). We can solve this system: multiplying the third equation in the system above by $q_i^-/q_{i+1}^+$, the fourth by $q_i^- q_{i+1}^-/q_{i+1}^+ q_{i+2}^+$, etc. and add these equations. We get the equation (4.25). Since $Y_i(t)$ is an increasing function, it might hit $Y_{k_0+1}(0) = x_{k_0+1}$ before the time horizon $T$. (If it does not, there is nothing else to prove.) Then $\tau_1$ is this hitting moment. The set-valued function $I$ is constant on $[0, \tau_1)$ but jumps at $\tau_1$. Using the memoryless property and induction, we repeat this proof starting from $\tau_1$ time instead of 0. Since the function $I$ increases at every $\tau_l$, and it can take set values which contain $\{i\}$ and are contained in $\{i, \dots, N\}$, there will be at most $N+1$ induction steps. $\square$

### *4.4.5 Proof of Theorem* 4.2.1

Take driving functions as in (4.10) satisfying (4.11). Let $\tau_0 := 0, \tau_1, \dots, \tau_K := T$ be the sequence of moments described in Lemma 4.4.9, and let $\overline{\tau}_0 := 0, \overline{\tau}_1, \dots, \overline{\tau}_{\overline{K}} := T$ be the corresponding sequence of moments for the driving function $\overline{X}$ instead of $X$. Arrange all these moments in the increasing order:

$$\rho_0 := 0 < \rho_1 := \tau_1 \wedge \overline{\tau}_1 < \rho_2 < \ldots < \rho_M := T.$$

Then it suffcies to show the theorem for $t \le \rho_1$. Indeed, suppose that we prove this, then we can use the memoryless property for Skorohod problems and prove this for $\rho_1 \le t \le \rho_2$, then for $\rho_2 \le t \le \rho_3$, etc. Extending the result from $[0, \rho_1]$ to $[0, T]$ requires reasoning analogous to the argument in proof of Lemma (4.4.8).

On $[0, \rho_1]$, we know explicit expressions for $Z$, $\overline{Z}$, $L$ and $\overline{L}$ from Lemma 4.4.9. Let $I(t)$ be the set-valued function defined in Lemma 4.4.9, and $\overline{I}(t)$ be the same function for $\overline{X}$ instead of $X$. Consider the following cases.

*Case 1.* $0 \le \alpha \le \overline{\alpha}$. Then $Z(t) \equiv X(t)$, $\overline{Z}(t) \equiv \overline{X}(t)$, $L(t) \equiv \overline{L}(t) \equiv 0$, and the statement is obvious.

*Case 2.* $\alpha \le 0 \le \overline{\alpha}$. Then $Z$ is nonincreasing, $\overline{Z} = \overline{X}$ is nondecreasing, $\overline{L}(t) \equiv 0$, and $L$ is nondecreasing. The statement follows trivially.

*Case 3.* $\alpha \le \overline{\alpha} \le 0$, and $x_i > 0$. Since $x \le \overline{x}$, we have: $\overline{x}_i > 0$, and the rest is exactly as in Case 1.

*Case 4.* $\alpha \le \overline{\alpha} \le 0$, and $x_i = 0$, $\overline{x}_i > 0$. Then $I(0) \supsetneq \overline{I}(0)$, and on $[0, \rho_1)$ we have: $\overline{L}(t) \equiv 0$, $L$ is nondecreasing, so

$$L(t) - L(s) \ge \overline{L}(t) - \overline{L}(s),\ 0 \le s \le t \le \rho_1.$$

Furthermore, $\overline{Z}(t) \equiv \overline{X}(t) = \overline{x}_i + \overline{\alpha} e_i t$. And $Z(t)$ is given by:

$$Z_j(t) = 0 \le \overline{Z}_j(t),\ j \in I(0);$$

$Z(t)$ is nonincreasing, so for $j \notin I(0)$ we have: $\overline{Z}_j(t) = \overline{Z}_j(0) = \text{const}$. Thus,

$$Z_j(t) \le Z_j(0) \le \overline{Z}_j(0) = \overline{Z}_j(t).$$

*Case 5.* $\alpha \le \overline{\alpha} \le 0$, and $x_i = \overline{x}_i = 0$. This is the most difficult case. We again have: $\overline{I}(0) \subseteq I(0)$, and on $[0, \rho_1]$ we get:

*Case 5.1.* $j \in \overline{I}(0)$, $Z_j(t) \equiv \overline{Z}_j(t) \equiv 0$, so trivially

$$Z_j(t) - Z_j(s) \le \overline{Z}_j(t) - \overline{Z}_j(s), \quad 0 \le s \le t.$$

Furthermore,

$$[\overline{L}(t)]_{\overline{I}(0)} = |\overline{\alpha}|[\overline{R}]^{-1}_{\overline{I}(0)}[e_i]_{\overline{I}(0)}t, \quad [L(t)]_{I(0)} = |\alpha|[R]^{-1}_{I(0)}[e_i]_{I(0)}t.$$

Applying Lemma 4.6.6 to $J = I(0)$, we get that $[R]_{I(0)}$ is a reflection nonsingular $\mathcal{M}$-matrix. Applying Lemma 4.6.1 to $[R]_{I(0)}$ instead of $R$ and $J = \overline{I}(0)$, we get:

$$[R]^{-1}_{\overline{I}(0)} \le [[R]^{-1}_{I(0)}]_{\overline{I}(0)}.$$

Also, $[e_i]_{\overline{I}(0)} = [[e_i]_{I(0)}]_{\overline{I}(0)} \ge 0$, and $|\overline{\alpha}| \le |\alpha|$. Since $R \le \overline{R}$, we have: $[R]_{\overline{I}(0)} \le [\overline{R}]_{\overline{I}(0)}$. Both $[R]_{\overline{I}(0)}$ and $[\overline{R}]_{\overline{I}(0)}$ are reflection nonsingular $\mathcal{M}$-matrices of the same size, so by Lemma 4.6.5 we have:

$$[R]^{-1}_{\overline{I}(0)} \ge [\overline{R}]^{-1}_{\overline{I}(0)} \ge 0. \tag{4.26}$$

In addition, by Lemma 4.6.4 we have:

$$\left[[R]^{-1}_{I(0)}[e_i]_{I(0)}\right]_{\overline{I}(0)} \ge \left[[R]^{-1}_{I(0)}\right]_{\overline{I}(0)} \left[[e_i]_{I(0)}\right]_{\overline{I}(0)} \tag{4.27}$$

Combining (4.26), (4.27) and the fact that $|\overline{\alpha}| \le |\alpha|$, we get: for $0 \le s \le t \le \rho_1$,

$$\begin{aligned}
[L(t)]_{\overline{I}(0)} - [L(s)]_{\overline{I}(0)} &= |\alpha| \left[[R]^{-1}_{I(0)}[e_i]_{I(0)}\right]_{\overline{I}(0)} (t-s) \ge |\alpha| \left[[R]^{-1}_{I(0)}\right]_{\overline{I}(0)} \left[[e_i]_{I(0)}\right]_{\overline{I}(0)} (t-s) \\
&\ge |\overline{\alpha}|[R]^{-1}_{\overline{I}(0)}[e_i]_{\overline{I}(0)}(t-s) \ge |\overline{\alpha}|[R]^{-1}_{\overline{I}(0)}[e_i]_{\overline{I}(0)}(t-s) = [\overline{L}(t)]_{\overline{I}(0)} - [\overline{L}(s)]_{\overline{I}(0)}.
\end{aligned}$$

In other words, for $j \in \overline{I}(0)$,

$$L_j(t) - L_j(s) \ge \overline{L}_j(t) - \overline{L}_j(s), \quad 0 \le s \le t \le \rho_1.$$

*Case 5.2.* $j \in I(0) \setminus \overline{I}(0)$. Then $Z_j(t) = 0 \le \overline{Z}_j(t)$. Now, $L_j$ is always nondecreasing, and $\overline{Z}_j > 0$, so $\overline{L}_j \equiv 0$. Thus,

$$L_j(t) - L_j(s) \ge 0 = \overline{L}_j(t) - \overline{L}_j(s), \ 0 \le s \le t \le \rho_1.$$

*Case 5.3.* $j \notin \overline{I}(0)$. Then $j \notin I(0)$. Let

$$I^c(0) := \{1, \ldots, d\} \setminus I(0), \quad \overline{I}^c(0) := \{1, \ldots, d\} \setminus \overline{I}(0).$$

The components of $Z$ as $\overline{Z}$ corresponding to the index set $\overline{I}^c(0)$ have the following dynamics:

$$\begin{cases}[Z(t)]_{I^c(0)} = [Z(0)]_{I^c(0)} + |\alpha|[R]_{I^c(0)I(0)}[R]^{-1}_{I(0)}[e_i]_{I(0)}t,\\ [\overline{Z}(t)]_{\overline{I}^c(0)} = [\overline{Z}(0)]_{\overline{I}^c(0)} + |\overline{\alpha}|[\overline{R}]_{\overline{I}^c(0)\overline{I}(0)}[\overline{R}]^{-1}_{\overline{I}(0)}[e_i]_{\overline{I}(0)}t.\end{cases}$$

Since $R$ and $\overline{R}$ are reflection nonsingular $\mathcal{M}$-matrices, and $R \le \overline{R}$, we have:

$$r_{ij} \le \overline{r}_{ij} \le 0, \; i \ne j. \tag{4.28}$$

In particular, this is true for $i \in I^c(0),\; j \in I(0)$, as well as for $i \in \overline{I}^c(0),\; j \in \overline{I}(0)$. But $I(0) \supseteq \overline{I}(0)$, and so $I^c(0) \subseteq \overline{I}^c(0)$. Therefore,

$$\begin{aligned}[\overline{Z}(t)]_{I^c(0)} &= [\overline{Z}(0)]_{I^c(0)} + |\overline{\alpha}|t\left[[\overline{R}]_{\overline{I}^c(0)\overline{I}(0)}[\overline{R}]^{-1}_{\overline{I}(0)}[e_i]_{\overline{I}(0)}\right]_{I^c(0)}\\ &= [\overline{Z}(0)]_{I^c(0)} - |\overline{\alpha}|t\left[[-\overline{R}]_{\overline{I}^c(0)\overline{I}(0)}[\overline{R}]^{-1}_{\overline{I}(0)}[e_i]_{\overline{I}(0)}\right]_{I^c(0)}.\end{aligned}$$

It follows from (4.28) that

$$0 \le [-\overline{R}]_{\overline{I}^c(0)\overline{I}(0)} \le [-R]_{\overline{I}^c(0)\overline{I}(0)}. \tag{4.29}$$

Also, $[\overline{R}]^{-1}_{\overline{I}(0)} \ge 0$, $[e_i]_{\overline{I}(0)} \ge 0$. By Lemma 4.6.3,

$$\left[[-\overline{R}]_{\overline{I}^c(0)\overline{I}(0)}[\overline{R}]^{-1}_{\overline{I}(0)}[e_i]_{\overline{I}(0)}\right]_{I^c(0)} = [-\overline{R}]_{I^c(0)\overline{I}(0)}[\overline{R}]^{-1}_{\overline{I}(0)}[e_i]_{\overline{I}(0)}. \tag{4.30}$$

By Lemma 4.6.7 and inequalities (4.29) and (4.30),

$$[-\overline{R}]_{I^c(0)\overline{I}(0)}[\overline{R}]^{-1}_{\overline{I}(0)}[e_i]_{\overline{I}(0)} \le [-R]_{I^c(0)\overline{I}(0)}[R]^{-1}_{\overline{I}(0)}[e_i]_{\overline{I}(0)}. \tag{4.31}$$

Since $\overline{I}(0) \subseteq I(0)$, applying Lemma 4.6.1, we get:

$$0 \le [R]^{-1}_{\overline{I}(0)} \le \left[[R]^{-1}_{I(0)}\right]_{\overline{I}(0)}, \tag{4.32}$$

Therefore, applying Lemma 4.6.7 again, and using that $[e_i]_{\overline{I}(0)} = \left[[e_i]_{I(0)}\right]_{\overline{I}(0)}$, we have:

$$[-R]_{I^c(0)\overline{I}(0)}[R]^{-1}_{\overline{I}(0)}[e_i]_{\overline{I}(0)} \le [-R]_{I^c(0)\overline{I}(0)}\left[[R]^{-1}_{I(0)}\right]_{\overline{I}(0)}\left[[e_i]_{I(0)}\right]_{\overline{I}(0)}. \tag{4.33}$$

By Lemma 4.6.2 (applied twice)

$$[-R]_{I^c(0)\overline{I}(0)}\left[[R]^{-1}_{I(0)}\right]_{\overline{I}(0)}\left[[e_i]_{I(0)}\right]_{\overline{I}(0)} \leq [-R]_{I^c(0)I(0)}[R]^{-1}_{I(0)}[e_i]_{I(0)}. \tag{4.34}$$

Combining (4.31), (4.32), (4.33) and (4.34), we get:

$$0 \leq \left[[-\overline{R}]_{\overline{I}^c(0)\overline{I}(0)}[\overline{R}]^{-1}_{\overline{I}(0)}[e_i]_{\overline{I}(0)}\right]_{I^c(0)} \leq [-R]_{I^c(0)I(0)}[R]^{-1}_{I(0)}[e_i]_{I(0)}.$$

But we also have: $|\alpha| \geq |\overline{\alpha}| \geq 0$. So

$$0 \leq \left[[-\overline{R}]_{\overline{I}^c(0)\overline{I}(0)}[\overline{R}]^{-1}_{\overline{I}(0)}[e_i]_{\overline{I}(0)}\right]_{I^c(0)}|\overline{\alpha}|t \leq [-R]_{I^c(0)I(0)}[R]^{-1}_{I(0)}[e_i]_{I(0)}|\alpha|t.$$

Finally, we get:

$$\begin{aligned}[\overline{Z}(t)]_{I^c(0)} &\geq [\overline{Z}(0)]_{I^c(0)} - |\alpha|t[-R]_{I^c(0)I(0)}[R]^{-1}_{I(0)}[e_i]_{I(0)} \\ &\geq [Z(0)]_{I^c(0)} + |\alpha|t[R]_{I^c(0)I(0)}[R]^{-1}_{I(0)}[e_i]_I = [Z(t)]_{I^c(0)}.\end{aligned}$$

So for $j \in I^c(0)$ we get:

$$0 \leq Z_j(t) \leq \overline{Z}_j(t), \quad t \in [0, \rho_1].$$

Finally, since $Z_j(t) > 0$ and $\overline{Z}_j(t) > 0$ for $t \in [0, \rho_1)$, we have: $L_j = \overline{L}_j = 0$ on this interval, and trivially

$$L_j(t) - L_j(s) \geq \overline{L}_j(t) - \overline{L}_j(s),\ 0 \leq s \leq t \leq \rho_1.$$

The proof is complete.

### *4.4.6 Proof of Theorem* 4.2.2.

As in the previous subsection, it suffices to prove the theorem for coupled regular linear driving functions

$$X(t) = x + \alpha e_i t, \quad \overline{X}(t) = \overline{x} + \overline{\alpha} e_i t,$$

which satisfy the conditions of Theorem 4.2.2. This means that $x \leq \overline{x}$, and $\alpha \leq \overline{\alpha}$. Assume the converse: there exist $t \in [0, T]$ and $j = 1, \ldots, N$ such that $Y_j(t) > \overline{Y}_j(t)$. Since $Y_k(0) \leq \overline{Y}_k(0)$, $k = 1, \ldots, N$, we can let

$$\tau_0 := \inf\{t \geq 0 \mid \exists j = 1, \ldots, N : Y_j(t) > \overline{Y}_j(t)\}.$$

In other words,

$$Y_k(\tau_0) \le \overline{Y}_k(\tau_0),\ \ k = 1, \ldots, N,$$

but there exists $j = 1, \ldots, N$ such that for every $\varepsilon > 0$ there exists $t \in (\tau_0, \tau_0 + \varepsilon)$ such that $Y_j(t) > \overline{Y}_j(t)$. W.l.o.g. by memoryless property, assume $\tau_0 = 0$. Then $Y_j(0) = \overline{Y}_j(0)$. Recall that $I(t) := \{k = i, \ldots, N \mid Y_k(t) = Y_i(t)\}$, and $\tau_1 := \inf\{t \ge 0 \mid I(t) \ne I(0)\} \wedge T$. Define $\overline{I}(t)$ and $\overline{\tau}_1$ similarly for $\overline{Y}$ in place of $Y$. Let $\varepsilon := \tau_1 \wedge \overline{\tau}_1$.

*Case 1.* $\alpha \le 0 \le \overline{\alpha}$. Then $Y_j$ are nonincreasing (follows from Lemma 4.4.11 and Lemma 4.4.10), $\overline{Y}_j$ are nondecreasing, and the statement is trivial.

*Case 2.* $\alpha \le \overline{\alpha} \le 0$. This can be reduced to Case 3 by Lemma 4.4.10.

*Case 3.* $0 \le \alpha \le \overline{\alpha}$. If $j < i$, then particles $Y_j$ and $\overline{Y}_j$ lie below $Y_i(0) = \overline{Y}_i(0)$ and therefore $Y_j(t) = \overline{Y}_j(t) = \text{const}$ on $[0, \varepsilon]$. Now, if $j \ge i$. Suppose $j \notin I(0)$, that is, $Y_j(0) > Y_i(0)$. Then, again, the particle $Y_j$ is unaffected by $Y_i$ moving upward, at least not until $Y_i$ hits $Y_j$, that is, not until $\tau_1 \wedge \overline{\tau}_1$. But the particle $\overline{Y}_j$ is nondecreasing, according to Lemma 4.4.11, so $\overline{Y}_j(t) \ge Y_j(t)$ on $[0, \varepsilon)$.

Therefore, we are left with the case $j \in I(0)$. Equivalently, $Y_j(0) = Y_i(0)$. And $Y_j(0) = \overline{Y}_j(0) \ge \overline{Y}_i(0)$, so $Y_i(0) \ge \overline{Y}_i(0)$. But by the conditions of the theorem, $Y_i(0) \le \overline{Y}_i(0)$, so $Y_i(0) = \overline{Y}_i(0)$. Thus,

$$Y_i(0) = \overline{Y}_i(0) = Y_j(0) = \overline{Y}_j(0),$$

and $j \in I(0) \cap \overline{I}(0)$. However, $I(0 \supseteq \overline{I}(0)$, because if $k \in \overline{I}(0)$, then $k \ge i$ and

$$Y_k(0) \le \overline{Y}_k(0) = \overline{Y}_i(0) = Y_i(0) \le Y_k(0),$$

so $Y_k(0) = Y_i(0)$, and $k \in I(0)$. Let $\overline{I}(0) = \{i, \ldots, \overline{k}_0\}$, and $I(0) = \{i, \ldots, k_0\}$. From $\overline{I}(0) \subseteq I(0)$ it follows that $\overline{k}_0 \le k_0$. Therefore, for $t \in [0, \varepsilon]$ we have:

$$Y_i(t) \equiv Y_j(t) = Y_i(0) + \alpha t \left[1 + \frac{q_i^-}{q_{i+1}^+} + \frac{q_i^- q_{i+1}^-}{q_{i+1}^+ q_{i+2}^+} + \ldots + \frac{q_i^- q_{i+1}^- \cdots q_{k_0-1}^-}{q_{i+1}^+ q_{i+2}^+ \cdots q_{k_0}^+}\right],$$

$$\overline{Y}_i(t) \equiv \overline{Y}_j(t) = \overline{Y}_i(0) + \overline{\alpha} t \left[1 + \frac{\overline{q}_i^-}{\overline{q}_{i+1}^+} + \frac{\overline{q}_i^- \overline{q}_{i+1}^-}{\overline{q}_{i+1}^+ \overline{q}_{i+2}^+} + \ldots + \frac{\overline{q}_i^- \overline{q}_{i+1}^- \cdots \overline{q}_{\overline{k}_0-1}^-}{\overline{q}_{i+1}^+ \overline{q}_{i+2}^+ \cdots \overline{q}_{\overline{k}_0}^+}\right].$$

But

$$\overline{q}_k^+ \ge q_k^+, \quad \overline{q}_k^- \le q_k^-, \ k = 1, \ldots, N; \ \overline{k}_0 \le k_0,$$

so

$$1 + \frac{q_i^-}{q_{i+1}^+} + \frac{q_i^- q_{i+1}^-}{q_{i+1}^+ q_{i+2}^+} + \ldots + \frac{q_i^- q_{i+1}^- \cdots q_{k_0-1}^-}{q_{i+1}^+ q_{i+2}^+ \cdots q_{k_0}^+} \ge 1 + \frac{\overline{q}_i^-}{\overline{q}_{i+1}^+} + \frac{\overline{q}_i^- \overline{q}_{i+1}^-}{\overline{q}_{i+1}^+ \overline{q}_{i+2}^+} + \ldots + \frac{\overline{q}_i^- \overline{q}_{i+1}^- \cdots \overline{q}_{\overline{k}_0-1}^-}{\overline{q}_{i+1}^+ \overline{q}_{i+2}^+ \cdots \overline{q}_{\overline{k}_0}^+}.$$

And $\alpha \le \overline{\alpha}$ and $Y_i(0) = \overline{Y}_i(0)$, we have: $Y_i(t) \le \overline{Y}_i(t)$ for $t \in [0, \varepsilon]$, which contradicts our assumption. This completes the proof of Theorem 4.2.2.

## *4.5 The case of totally asymmetric collisions*

So far we considered the case when the collisions between particles are either symmetric (the local time of collision is split evenly between the particles) or asymmetric but not totally asymmetric (the local time is split not evenly, but both particles receive a certain share of the local time). We would like to consider totally asymmetric collisions, when all of the local time is received by only one particle, and the other particle does not experience any influence of a collision. In other words, when one particle reflects on the other. Similar systems were considered in [38]; they are related to random matrices and random surfaces.

Suppose the particles with ranks $k$ and $k+1$ collide. Then the share of the local time of collision received by the $k$th particle is $q_k^-$, and the share received by the $k+1$st particle is $q_{k+1}^+$, where $q_{k+1}^+ + q_k^- = 1, \ q_{k+1}^+, \ q_k^- \ge 0$. So far we considered the case when these quantities are strictly positive. Now we allow the possibility that one of them equals zero. Does the system exist in this case? Consider finite systems of competing functions with asymmetric collisions.

**Theorem 4.5.1.** *(i) The matrix $R$ is completely-$\mathcal{S}$ if and only if there do not exist $1 \le k \le l \le N$ such that $q_k^+ = q_l^- = 1$.*

*(ii) In this case, for every continuous driving function there exists a unique system of competing functions with given continuous driving terms and parameters of collisions $(q_k^\pm)_{1 \le k \le N}$.*

*Proof.* (i) The matrix $R$ is completely-$\mathcal{S}$ if and only if the matrix $Q := I_{N-1} - R = (q_{ij})_{1\le i,j\le N-1} \ge 0$ has spectral radius strictly less than one. If there do not exist $1 \le k \le l \le N$ such that $q_k^+ = q_l^- = 1$, then we can consider $Q^T$ as a substochastic matrix with one of row sums strictly less than one. Similarly to [71, Section 2.1], we conclude that it has spectral raduis strictly less than one. If there exist $1 \le k \le l \le N$ such that $q_k^+ = q_l^- = 1$, then consider the principal submatrix $\tilde{Q} = (q_{ij})_{k-1\le i,j\le l}$. It is easy to see that each column sum of $\tilde{Q}$ is one, so $\tilde{Q}'\mathbf{1} = \mathbf{1}$, and $Q^T v = v$, where $v = (v_j)_{1\le j\le N-1},\ v_j = 1$ if $k-1 \le j \le l$ and $v_j = 0$ otherwise. So 1 is an eigenvalue of $Q^T$, and therefore of $Q$. This implies that $R$ is not completely-$\mathcal{S}$.

(ii) Let $X$ be the driving function. Let $Z$ be the solution of the Skorohod problem with driving function $(X_2 - X_1, \ldots, X_N - X_{N-1})'$ and reflection matrix $R$. Then $Z$ is the gap process for the would-be system $Y$ of competing functions with the given parameters of collision and driving function $X$. Now, suppose $k_0$ is the minimal $k = 1, \ldots, N-1$ such that $q_{k_0}^+ = 0$. If there is no such $k$ then let $k_0 = N$. Let $\alpha_1, \alpha_2, \ldots, \alpha_{k_0}$ be defined by (3.10), and $\alpha_{k_0+1} = \ldots = \alpha_N = 0$. The rest of the proof goes as in Theorem 3.4.3. $\square$

## 4.6 Appendix: Technical Lemmata

**Lemma 4.6.1.** *Take a $d \times d$-reflection nonsingular $\mathcal{M}$-matrix $R$ and fix a nonempty subset $J \subseteq \{1, \ldots, d\}$. Then*

$$0 \le [R]_J^{-1} \le [R^{-1}]_J.$$

*Proof.* Since $R = I_d - Q$, where $Q \ge 0$ is a $d \times d$-matrix with spectral radius strictly less than one, we can apply the Neumann series:

$$R^{-1} = I_d + Q + Q^2 + \ldots \tag{4.35}$$

By Lemma 4.6.6, $[R]_J = I_{|J|} - [Q]_J$ is also a reflection nonsingular $\mathcal{M}$-matrix, so we have:

$$[R]_J^{-1} = I_{|J|} + [Q]_J + [Q]_J^2 + \ldots$$

But from (4.35) we get:

$$[R^{-1}]_J = I_{|J|} + [Q]_J + [Q^2]_J + \dots$$

Let us show that $[Q^k]_J \geq [Q]_J^k$ for $k = 1, 2, 3, \dots$. This can be proved by induction using Lemma 4.6.2. $\square$

**Lemma 4.6.2.** *Take nonnegative matrices $A$ ($m \times d$) and $B$ ($d \times n$), and let $I \subseteq \{1, \dots, m\}$, $J \subseteq \{1, \dots, d\}$, $K \subseteq \{1, \dots, n\}$ be nonempty subsets. Then*

$$[A]_{IJ}[B]_{JK} \leq [AB]_{IK}.$$

*Proof.* Let $A = (a_{ij})$ and $B = (b_{ij})$. Then for $i \in I$ and $k \in K$,

$$([A]_{IJ}[B]_{JK})_{ik} = \sum_{j \in J} a_{ij} b_{jk} \leq \sum_{i=1}^{d} a_{ij} b_{jk} = (AB)_{ik} = ([AB]_{IK})_{ik}.$$

$\square$

**Lemma 4.6.3.** *Take a $d \times n$-matrix $A$ and a vector $a \in \mathbb{R}^n$. Let $I \subseteq \{1, \dots, d\}$ be a nonempty subset. Then $[Aa]_I = [A]_{I \times \{1,\dots,n\}} a$.*

The proof is trivial.

**Lemma 4.6.4.** *Take a $d \times d$-nonnegative matrix $A$ and a nonnegative vector $a \in \mathbb{R}^d$. Let $J \subseteq \{1, \dots, d\}$ be a nonempty subset. Then $[Aa]_J \geq [A]_J [a]_J$.*

The proof is trivial.

**Lemma 4.6.5.** *Let $R \leq \overline{R}$ be two $d \times d$-reflection nonsingular $\mathcal{M}$-matrices. Then $R^{-1} \geq \overline{R}^{-1} \geq 0$.*

*Proof.* Apply Neumann series again: if

$$R = I_d - Q, \quad \overline{R} = I_d - \overline{Q},$$

then $\overline{Q} \geq Q \geq 0$, and so $\overline{Q}^k \geq Q^k \geq 0$, $k = 1, 2, \dots$. Thus,

$$R^{-1} = I_d + Q + Q^2 \geq I_d + \overline{Q} + \overline{Q}^2 + \dots = \overline{R}^{-1}.$$

$\square$

**Lemma 4.6.6.** *If $R$ is a $d \times d$-reflection nonsingular $\mathcal{M}$-matrix and $I \subseteq \{1, \ldots, d\}$ is a nonempty subset, then $[R]_I$ is also a reflection nonsingular $\mathcal{M}$-matrix.*

*Proof.* Use Lemma 2.2.1 from Chapter 2, which corresponds to [103, Lemma 2.1]. A $d \times d$-matrix $R = (r_{ij})$ is a reflection nonsingular $\mathcal{M}$-matrix if and only if

$$r_{ii} = 1,\ i = 1, \ldots, d;\ \ r_{ij} \le 0,\ i \ne j,$$

and, in addition, $R$ is completely-$\mathcal{S}$, which means that for every principal submatrix $[R]_J$ of $R$ there exists a vector $u > 0$ such that $[R]_J u > 0$. Now, switch from $R$ to $[R]_I$. The same conditions hold:

$$r_{ii} = 1,\ i \in I;\ \ r_{ij} \le 0,\ i \ne j,\ i, j \in I,$$

and, in addition, for every principal submatrix $[[R]_I]_J = [R]_J$ of $[R]_I$, where $J \subseteq I$, there exists a vector $u > 0$ such that $[R]_J u > 0$. This means that $[R]_I$ is also a reflection nonsingular $\mathcal{M}$-matrix. □

**Lemma 4.6.7.** *If $A \ge B \ge 0$ and $C \ge D \ge 0$ are matrices such that the matrix products $AC$ and $BD$ are well defined, then $AC \ge BD \ge 0$.*

The proof is trivial.

# Chapter 5

# TRIPLE AND SIMULTANEOUS COLLISIONS

In section 5.1, we state main results for systems of competing Brownian particles, both classical and with asymmetric collisions. In section 5.2, we state the main result for an SRBM in the orthant, and we prove it in section 5.3. The proof of results from section 5.1 is in section 5.4. Section 5.5 is an Appendix; it contains some technical proofs.

## *5.1 Results for Competing Brownian Particles: Theorems* 5.1.1 *and* 5.1.3

Now, let us define the two concepts: a *triple collision* and a *simultaneous collision*.

**Definition 22.** A *triple collision at time $t$* occurs if there exists a rank $k = 2, \ldots, N-1$ such that $Y_{k-1}(t) = Y_k(t) = Y_{k+1}(t)$.

A triple collision is sometimes an undesirable phenomenon. For example, existence and uniqueness of a strong solutions of the SDE (3.1) has been proved only up to the first moment of a triple collision, see [59, Theorem 2]. In this chapter, we give a necessary and sufficient condition for absence of triple collisions with probability one.

**Definition 23.** A *simultaneous collision* at time $t$ occurs if there are ranks $k \ne l$ such that such that $Y_k(t) = Y_{k+1}(t),\ Y_l(t) = Y_{l+1}(t)$.

Note that a triple collision is a particular case of a simultaneous collision. Let us state the main result of this chapter.

**Theorem 5.1.1.** *Consider a system from Definition* 12.

*(i) Suppose the sequence $(\sigma_n^2)_{1\le n\le N}$ is concave, that is,*

$$\sigma_{k+1}^2 - \sigma_k^2 \le \sigma_k^2 - \sigma_{k-1}^2, \quad k = 2, \ldots, N-1. \tag{5.1}$$

*Then, with probability one, there are no triple and no simultaneous collisions at any time* $t > 0$.

*(ii) If the condition* (5.1) *fails for a certain* $k = 2, \ldots, N-1$, *then with positive probability there exists a moment* $t > 0$ *such that there is a triple collision between particles with ranks* $k-1$, $k$, *and* $k+1$ *at time* $t$.

The proof of this result is given in Section 5.4. We can state a remarkable corollary of this theorem.

**Corollary 5.1.2.** *Take a system from Definition* 12. *Suppose a.s. there are no triple collisions at any moment* $t > 0$. *Then a.s. there are no simultaneous collisions at any moment* $t > 0$.

It is interesting that a system of $N = 4$ particles can have a.s. no simultaneous collisions of the form

$$Y_1(t) = Y_2(t), \quad Y_3(t) = Y_4(t), \tag{5.2}$$

and at the same time it can have triple collisions with positive probability. For example, if you take

$$\sigma_1 = \sigma_4 = 1, \quad \text{and} \quad \sigma_2 = \sigma_3 = 1 - \varepsilon \quad \text{for sufficiently small} \quad \varepsilon > 0,$$

then there are a.s. no simultaneous collisions of the form (5.2), but with positive probability there is a triple collision of ranked particles $Y_1$, $Y_2$, and $Y_3$, and with positive probability there is a triple collision of ranked particles $Y_2$, $Y_3$, and $Y_4$. Another example: if

$$\sigma_1 = \sigma_3 = 1, \quad \text{and} \quad \sigma_2 = \sigma_4 = 1 + \varepsilon \quad \text{for sufficiently small} \quad \varepsilon > 0,$$

then there are a.s. no simultaneous collisions of the form (5.2), and a.s. no triple collisions of ranked particles $Y_1$, $Y_2$, and $Y_3$, but with positive probability there is a triple collision of ranked particles $Y_2$, $Y_3$, and $Y_4$. This is shown in Chapter 6 (which corresponds to the paper [102, Subsection 1.2]).

We can also give a similar necessary and sufficient condition for the case of asymmetric collisions.

**Theorem 5.1.3.** *Consider a system of competing Brownian particles with asymmetric collisions from Definition* 14.

*(i) Suppose the following condition is true:*

$$(q^-_{k-1} + q^+_{k+1})\sigma_k^2 \ge q^-_k\sigma^2_{k+1} + q^+_k\sigma^2_{k-1}, \quad k = 2, \dots, N-1. \tag{5.3}$$

*Then, with probability one, there are no triple and no simultaneous collisions at any time* $t > 0$.

*(ii) If the condition* (5.3) *is violated for some* $k = 2, \dots, N-1$, *then with positive probability there exists a moment* $t > 0$ *such that there is a triple collision between particles with ranks* $k-1,\ k,$ *and* $k+1$ *at time* $t$.

Note that Theorem 5.1.1 is a particular case of this theorem for $q^{\pm}_k = 1/2,\ k = 1, \dots, N$. Corollary 5.1.2 is also true for systems with asymmetric collisions.

*Remark* 11. A system of competing Brownian particles has a simultaneous collision at time $t$ if and only if the gap process hits non-smooth parts of the boundary $\partial S$ at time $t$. This is our method of proof: we state and prove results for an SRBM, and then we translate them into the language of systems of competing Brownian particles.

## 5.2 Results for an SRBM in the Orthant: Theorem 5.2.1

In this subsection, we state a necessary and sufficient condition for an SRBM a.s. to avoid non-smooth parts of the boundary. For the rest of this subsection, fix $d \ge 2$. Suppose $R$ is a $d \times d$ reflection nonsingular $\mathcal{M}$-matrix. Fix a vector $\mu \in \mathbb{R}^d$ and a $d \times d$ positive definite symmetric matrix $A$. Recall the notation $S = \mathbb{R}^d_+$ and consider the process $Z = (Z(t), t \ge 0) = \mathrm{SRBM}^d(R, \mu, A)$, starting from some point $x \in S$.

Let us give a necessary and sufficient condition for an SRBM a.s. not hitting non-smooth parts of the boundary $\partial S$ of the orthant $S$.

**Theorem 5.2.1.** *(i) Suppose the following condition holds:*

$$r_{ij}a_{jj} + r_{ji}a_{ii} \ge 2a_{ij}, \quad 1 \le i, j \le d. \tag{5.4}$$

*Then with probability one, there does not exist $t > 0$ such that $Z$ hits non-smooth parts of the boundary at time $t$.*

*(ii) If the condition* (5.4) *is violated for some* $1 \le i < j \le d$*, then with positive probability there exists* $t > 0$ *such that* $Z_i(t) = Z_j(t) = 0$*.*

*Remark* 12. The condition (5.4) can be written in the matrix form as $RD + DR^T \ge 2A$, where $D = \mathrm{diag}(A) = \mathrm{diag}(a_{11}, \dots, a_{dd})$ is the diagonal $d \times d$-matrix with the same diagonal entries as $A$. The case when we have equality in (5.4) instead of inequality, is very important: the condition

$$RD + DR^T = 2A \quad \Leftrightarrow \quad r_{ij}a_{jj} + r_{ji}a_{ii} = 2a_{ij}, \quad 1 \le i, j \le d, \tag{5.5}$$

is precisely the *skew-symmetry condition*, see Introduction.

*Remark* 13. Whether an $\mathrm{SRBM}^d(R, \mu, A)$ a.s. avoids non-smooth parts of the boundary depends only on the matrices $R$ and $A$, not on the initial condition $Z(0)$ or the drift vector $\mu$. Some general results of this type are shown in subsection 3.2, Lemma 5.3.1. But the actual probability of hitting non-smooth parts of the boundary, if it is positive, does depend on $\mu$ and the initial condition, see Remark 15.

## 5.3 *Proof of Theorem* 5.2.1

### *5.3.1 Outline of the proof*

We can define a reflected Brownian motion not only in the orthant, but in more general domains: namely, in convex polyhedra, see [17]. Similarly to an SRBM in the orthant, this is a process which behaves as a Brownian motion in the interior of the domain and is reflected according to a certain vector at each face of the boundary. We can reduce an SRBM in the orthant with an arbitrary covariance matrix to a reflected Brownian motion in a convex polyhedron with identity covariance matrix. This construction is carried out in detail in subsection 3.5, Lemma 5.3.6.

Let us give a brief preview here. Consider an SRBM $Z = (Z(t), t \geq 0)$ in the orthant $\mathbb{R}^d_+$ with covariance matrix $A$. Consider the process

$$\overline{Z} = (\overline{Z}(t), t \geq 0), \quad \overline{Z}(t) = A^{-1/2}Z(t), \tag{5.6}$$

which is a reflected Brownian motion in the domain $A^{-1/2}\mathbb{R}^d_+ := \{A^{-1/2}z \mid z \in \mathbb{R}^d_+\}$ with identity covariance matrix.

For a reflected Brownian motion in a polyhedral domain with identity covariance matrix, a sufficient condition (the *skew-symmetry condition*) for a.s. not hitting non-smooth parts of the boundary is known, see [124, Theorem 1.1]. Note that there are two forms of the skew-symmetry condition. One is for an SRBM in the orthant with arbitrary covariance matrix, which is (5.5). The other is for a reflected Brownian motion in a convex polyhedron with identity covariance matrix, which was introduced in [124]; in this chapter, it is going to be given in (5.16). In Lemma 5.3.8 we prove that under this linear transformation (5.6), these two conditions match. This justifies why they bear the same name. This allows us (in Lemma 5.3.11) to prove part (i) of Theorem 5.2.1 under the skew-symmetry condition (5.5).

Now, we need to show this for a more general condition (5.4). We reduce this general case to the case of the skew-symmetry condition (5.5) by stochastic comparison (Lemma 4.3.5). We introduce an SRBM with new reflection matrix $\tilde{R}$ which satisfies the skew-symmetry condition and such that $\tilde{R} \geq R$.

To prove part (ii), we first consider the case $d = 2$. The domain $A^{-1/2}\mathbb{R}^2_+$ is in this case a *two-dimensional wedge*, which can be written in polar coordinates

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta,$$

as

$$0 \leq r < \infty, \quad \xi_2 \leq \theta \leq \xi_1,$$

where $\xi_1$, $\xi_2$ are angles such that $\xi_2 \leq \xi_1 \leq \xi_2 + \pi$. We mentioned that a reflected Brownian motion in this domain with zero drift vector and identity covariance matrix was studied in [116], [121], [122], [123]. For this process, hitting non-smooth parts of the boundary means hitting the corner of the wedge (the origin). The result [116, Theorem 2.2] gives a necessary and sufficient condition for a.s. avoiding the corner. Using the linear transformation (5.6), we can then translate these results for an SRBM in the positive quadrant with general covariance matrix. This proves (ii) for $d = 2$.

To prove Theorem 5.2.1 for the general $d$, we again use comparison techniques. We consider any two components $Z_i, Z_j$ of the process $Z = (Z(t), t \ge 0) = \mathrm{SRBM}^d(R, \mu, A)$, and compare them with a two-dimensional SRBM using comparison techniques from Chapter 4.

Some parts of the calculations in this proof below have been done in certain previous articles. For example, the linear transformation $z \mapsto A^{-1/2}z$ and the way it transforms an SRBM in the orthant have been studied in the following articles: [52, Section 9, Theorem 23] (general dimension, under the skew-symmetry condition); [71, Proposition 2] (dimension $d = 2$). However, to make the exposition as lucid and self-contained as possible, we decided to do all calculations from scratch.

*Remark* 14. In this artlce, we define a reflected Brownian motion in Definition 4 as a semimartingale. Similarly, in the article [17] a reflected Brownian motion in a convex polyhedron is defined in a semimartingale form; we present this in Definition 25. However, in the papers [116] and [124], a reflected Brownian motion is not given in a semimartingale form. Instead, it is defined as a solution to a certain submartingale problem: see Definition 26. We use the semimartingale definition, and in Lemma 5.3.5 we prove that the semimartingale form of a reflected Brownian motion also satisfies the submartingale definition. This shows that we can indeed use the results from [116] and [124].

### *5.3.2 Girsanov removal of drift and independence of the initial conditions*

In this subsection, fix $d \ge 2$. Let $R$ be a $d \times d$ reflection nonsingular $\mathcal{M}$-matrix. Let $A$ be a $d \times d$ symmetric positive definite matrix, and let $\mu \in \mathbb{R}^d$. For every $x \in S$, denote by $\mathbf{P}_x$ the probability measure corresponding to the $\mathrm{SRBM}^d(R, \mu, A)$ starting from $x$.

Consider a general edge $S_I$ on the boundary $\partial S$. For example, $S_{\{i,j\}} = S_i \cap S_j$ for $i \ne j$

is a piece of the non-smooth parts of the boundary $\partial S$. In this chapter, we are interested in an $\mathrm{SRBM}^d(R, \mu, A)$ hitting or avoiding these edges. But for this subsection, we shall work with a general edge $S_I$ of $S$.

The main result of this subsection is that the property of an SRBM to a.s. avoid $S_I$ is independent of the starting point $x \in S$ and of the drift vector $\mu$. The proof is postponed until the end of this subsection.

**Proposition 5.3.1.** *Let* $Z = (Z(t) \geq 0)$ *be an* $\mathrm{SRBM}^d(R, \mu, A)$. *Let*

$$p(x, R, \mu, A) = \mathbf{P}_x \left(\exists\ t > 0 :\ Z(t) \in S_I\right).$$

*Fix a* $d \times d$ *reflection nonsingular* $\mathcal{M}$*-matrix* $R$ *and a positive definite symmetric* $d \times d$ *matrix* $A$. *Then one of these two statements is true:*

- *For all* $\mu \in \mathbb{R}^d$ *and* $x \in S$, *we have:* $p(x, R, \mu, A) = 0$: *(the edge* $S_I$ *is avoided).*

- *For all* $\mu \in \mathbb{R}^d$ *and* $x \in S$, *we have:* $p(x, R, \mu, A) > 0$: *(the edge* $S_I$ *is hit).*

*Remark* 15. We can reformulate Lemma 5.3.1 as follows: whether an $\mathrm{SRBM}^d(R, \mu, A)$ hits the edge $S_I$ does not depend on the initial conditions and the drift vector $\mu$; it depends only on the reflection matrix $R$ and the covariance matrix $A$.

However, suppose $\mathrm{SRBM}^d(R, \mu, A)$ hits the edge $S_I$, so the probability $p(x, R, \mu, A)$ is positive. What is its exact value? This probability *does* depend on the drift vector $\mu$ and the initial condition $x \in S$. Let us give a one-dimensional example: a reflected Brownian motion on the positive half-line $\mathbb{R}_+$ with no drift. With probability one, it hits the origin (which is the same as hitting the edge $S_{\{1\}}$). But a reflected Brownian motion on $\mathbb{R}_+$ with positive drift $b$, starting from $x > 0$, hits the origin with probability $e^{-2bx}$, see [7, Part 2, Section 2, formula 2.0.2]. This does depend on the drift $b$ and the initial condition $x$.

**Definition 24.** We say that an $\mathrm{SRBM}^d(R, \mu, A)$ *avoids non-smooth parts of the boundary* $\partial S$ *of the orthant* $S$ if it avoids every edge $S_I$ with $|I| = 2$. Otherwise, we say that an $\mathrm{SRBM}^d(R, \mu, A)$ *hits non-smooth parts of the boundary* $\partial S$.

From the discussion just above, we see: the property of hitting non-smooth parts of the boundary is independent of the initial condition $x$ and of the drift vector $\mu$. It depends only on $R$ and $A$. We can also see it from Theorem 5.2.1: the condition (5.4) involves only elements of $R$ and $A$.

### *5.3.3 Proof of Proposition* 5.3.1

We split the proof of Lemma 5.3.1 in two steps. First, we show independence of a starting point $x \in S$ in Lemma 5.3.2, then of a drift vector $\mu \in \mathbb{R}^d$ in Lemma 5.3.3, using the Girsanov transformation.

**Lemma 5.3.2.** *For fixed parameters $R, \mu, A$ of an SRBM, we have: either $p(x, R, \mu, A) = 0$ for all $x \in S$, or $p(x, R, \mu, A) > 0$ for all $x \in S$. In other words, either an* $\mathrm{SRBM}^d(R, \mu, A)$ *hits the edge $S_I$, or it avoids the edge $S_I$.*

*Proof.* Since the family of the processes $Z = (Z(t), t \geq 0) = \mathrm{SRBM}^d(R, \mu, A)$, starting from different points $x \in S$, is Feller continuous, the function

$$f(z) := \mathbf{P}_z \left(\exists t > 0 : Z(t) \in S_I\right)$$

is continuous on $S$. Let $P^t(x, C) = \mathbf{P}_x(Z(t) \in C)$ be the transition function for the $\mathrm{SRBM}^d(R, \mu, A)$. By the Markov property,

$$\mathbf{P}_z \left(\exists t > 1 : Z(t) \in S_I\right) = \int_S P^1(z, dy) f(y). \tag{5.7}$$

But

$$\mathbf{P}_z \left(\exists t > 1 : \ Z(t) \in S_I\right) \leq \mathbf{P}_z \left(\exists t > 0 : \ Z(t) \in S_I\right) = f(z). \tag{5.8}$$

Combining (5.7) and (5.8), we have:

$$\int_S f(y) P^1(z, dy) \leq f(z).$$

Suppose for some $z_0 \in S$ we have: $f(z_0) > 0$. Since $f$ is continuous, there exists an open neighborhood $U$ of $z_0$ in $S$ such that $f(z) \geq f(z_0)/2 > 0$ for $z \in U$. But $U$ has positive

Lebesgue measure, and so $P^1(z, U) > 0$ for $z \in S$. Therefore, $f(z) \geq P^1(z, U)f(z_0)/2 > 0$ for all $z \in S$.

We have proved that if $f(z_0) > 0$ for at least one $z_0 \in S$, then $f(z) > 0$ for all $z \in S$. This completes the proof of the lemma. $\square$

**Lemma 5.3.3.** *Fix a nonempty subset* $I \subseteq \{1, \dots, d\}$. *Then an* $\mathrm{SRBM}^d(R, \mu, A)$ *avoids* $S_I$ *if and only if an* $\mathrm{SRBM}^d(R, 0, A)$ *avoids* $S_I$.

*Proof.* Using Lemma 5.3.2, without loss of generality, fix a starting point $z \in S$, the same for both processes. Let $Z = \mathrm{SRBM}^d(R, \mu, A)$, starting from $z$, and let $\overline{Z} = \mathrm{SRBM}^d(R, 0, A)$, starting from $z$. Let $P, \overline{P}$ be the distributions of the processes $Z, \overline{Z}$ on the space $C(\mathbb{R}_+, \mathbb{R}^d)$ of continuous functions $\mathbb{R}_+ \to \mathbb{R}^d$. For every $T > 0$, let $\mathcal{G}_T$ be the $\sigma$-subalgebra of the Borel $\sigma$-algebra of $C(\mathbb{R}_+, \mathbb{R}^d)$, generated by the values of $x(s),\ 0 \leq s \leq T$ for all functions $x \in C(\mathbb{R}_+, \mathbb{R}^d)$. By the Girsanov theorem, for every $T > 0$, the restrictions $P|_{\mathcal{G}_T}$ and $\overline{P}|_{\mathcal{G}_T}$ are mutually absolutely continuous: they have common events of probability one. Therefore, the following statements are equivalent:

- With probability 1, there is no $t \in (0, T]$ such that $Z(t) \in S_I$;
- With probability 1, there is no $t \in (0, T]$ such that $\overline{Z}(t) \in S_I$.

Suppose that with probability 1, there is no $t > 0$ such that $Z_i(t) = 0$ for each $i \in I$; then for every $T > 0$, with probability 1, there is no $t \in (0, T]$ such that $\overline{Z}_i(t) = 0$. Since $T > 0$ is arbitrary, we have: with probability 1, there is no $t > 0$ such that $\overline{Z}_i(t) = 0$ for each $i \in I$. The converse statement is proved similarly. $\square$

### *5.3.4 An SRBM in a convex polyhedron*

Let us give a definition of an SRBM in convex polyhedra from [17]. Fix the dimension $d \geq 1$. First, let us define the state space, a polyhedral domain $\mathcal{P} \subseteq \mathbb{R}^d$. Fix $m \geq 1$, the number of edges. Let $n_1, \dots, n_m \in \mathbb{R}^d$ be unit vectors, and let $b_1, \dots, b_m \in \mathbb{R}$. The domain $\mathcal{P}$ is defined by

$$\mathcal{P} := \{x \in \mathbb{R}^d \mid n_i \cdot x \geq b_i, \quad i = 1, \dots, m\}. \tag{5.9}$$

We assume that the interior of $\mathcal{P}$ is nonempty and for each $j = 1, \dots, m$ we have:

$$\{x \in \mathbb{R}^d \mid n_i \cdot x \geq b_i, \quad i = 1, \dots, m, \ i \neq j\} \neq \mathcal{P}. \tag{5.10}$$

In this case, the *edges* of $\mathcal{P}$:

$$\mathcal{P}_i = \{x \in \mathcal{P} \mid n_i \cdot x = b_i\}, \quad i = 1, \dots, m,$$

are $(d-1)$-dimensional. Note that the vectors $n_i$, $i = 1, \dots, m$, are inward unit normal vectors to each of the faces $\mathcal{P}_1, \dots, \mathcal{P}_m$. Now, let us define an SRBM in the domain $\mathcal{P}$. Fix the parameters of this SRBM: a vector $\mu \in \mathbb{R}^d$, a $d \times d$ positive definite symmetric matrix $A$ and a $d \times m$-matrix $R$.

**Definition 25.** Fix a starting point $x \in \mathcal{P}$. Take $B = (B(t), t \geq 0)$ to be a $d$-dimensional Brownian motion with drift vector $\mu$ and covariance matrix $A$, starting from $x$. Take an adapted continuous $\mathcal{P}$-valued process $Z = (Z(t), t \geq 0)$ and an adapted continuous $\mathbb{R}^m$-valued process

$$L = (L(t), t \geq 0), \quad L(t) = (L_1(t), \dots, L_m(t))',$$

such that:

(i) $Z(t) = B(t) + RL(t), \quad t \geq 0$;

(ii) for every $i = 1, \dots, m$, $L_i(0) = 0$, $L_i$ is nondecreasing and can increase only when $Z(t) \in \mathcal{P}_i$.

The process $Z$ is called a *semimartingale reflected Brownian motion* (SRBM) in the domain $\mathcal{P}$ with *reflection matrix* $R$, *drift vector* $\mu$ and *covariance matrix* $A$. This process is denoted by $\mathrm{SRBM}^d(\mathcal{P}, R, \mu, A)$.

*Remark* 16. A particular case is an SRBM in the orthant $S$, which was introduced in Section 2: $\mathrm{SRBM}^d(R, \mu, A)$ is the same as $\mathrm{SRBM}^d(S, R, \mu, A)$.

Let $v_i$ be the $i$th column of $R$. An $\mathrm{SRBM}^d(\mathcal{P}, R, \mu, A)$ behaves as a $d$-dimensional Brownian motion with drift vector $\mu$ and covariance matrix $A$ inside $\mathcal{P}$. On each face $\mathcal{P}_i$, it is reflected in the direction of the vector $v_i$.

The paper [17] contains an existence and uniqueness result for an SRBM in $\mathcal{P}$. We present this result in a slightly weaker version, which is still sufficient for our purposes. For any nonempty subset $I \subseteq \{1, \ldots, m\}$, let $\mathcal{P}_I := \cap_{i\in I}\mathcal{P}_i$. A *positive linear combination* of vectors $u_1, \ldots, u_q$ is any vector $\alpha_1 u_1 + \ldots + \alpha_q u_q$ with $\alpha_1, \ldots, \alpha_q > 0$.

**Assumption 1.** For every nonempty subset $I \subseteq \{1, \ldots, m\}$, we have:

(i) $\mathcal{P}_I \neq \varnothing$ and $\mathcal{P}_J \subsetneq \mathcal{P}_I$ for $I \subsetneq J \subseteq \{1, \ldots, m\}$;

(ii) there is a positive linear combination $v$ of vectors $v_i,\ i \in I$, such that $v \cdot n_i > 0,\ i \in I$;

(iii) there is a positve linear combination $n$ of vectors $n_i,\ i \in I$, such that $n \cdot v_i > 0,\ i \in I$.

The following result in an immediate corollary of [17, Theorem 1.3].

**Proposition 5.3.4.** *Under Assumption* 1*, for every $x \in \mathcal{P}$ there exists in the weak sense the process*

$$Z^{(x)} = (Z^{(x)}(t), t \geq 0) = \mathrm{SRBM}^d(\mathcal{P}, R, \mu, A),$$

*starting from $Z^{(x)}(0) = x$, and it is unique in law. This family of processes $(Z^{(x)}, x \in \mathcal{P})$ is Feller continuous strong Markov.*

*Remark* 17. By Assumption 1(ii) applied to a subset $I = \{i\}$, we have: $v_i \cdot n_i > 0$. So we can normalize $v_i$ to make $v_i \cdot n_i = 1$. This is done by replacing $v_i$ by $k_i v_i$ for $k_i := (v_i \cdot n_i)^{-1}$ and replacing $L_i$ by $k_i^{-1} L_i$. Doing this for each $i = 1, \ldots, m$ is called *standard normalization.* The new reflection matrix is $\overline{R} = R\mathcal{D}$, where $\mathcal{D} = \mathrm{diag}((v_1 \cdot n_1)^{-1}, \ldots, (v_m \cdot n_m)^{-1})$. If $\overline{v}_i = k_i v_i$ is the $i$th column of $\overline{R}$, we can decompose it into the sum

$$\overline{v}_i = n_i + q_i, \tag{5.11}$$

where

$$q_i \cdot n_i = (\overline{v}_i - n_i) \cdot n_i = \overline{v}_i \cdot n_i - n_i \cdot n_i = 1 - 1 = 0, \quad i = 1, \ldots, m.$$

These vectors $n_i$ and $q_i$ are called the *normal and tangential components* of the reflection vector $\overline{v}_i$, respectively. Similar normalization was done for an SRBM in the orthant in [8, Appendix B].

As mentioned above, in the papers [116], [121], [122], [124], [123], reflected Brownian motion was defined as a solution to a certain submartingale problem. We are going to show that if an SRBM is defined in a semimartingale form, as in Definition 25, then it is also a solution to this submartingale problem, so we can use the results of the papers mentioned above.

**Definition 26.** Take a convex polyhedron $\mathcal{P}$ from (5.9) and the parameters $R, \mu, A$ from Definition 25. The symbol $C^2_c(\mathcal{P})$ stands for the family of twice continuously differentiable functions $f : \mathcal{P} \to \mathbb{R}$ with compact support. Define the following operator for functions $f \in C^2_c(\mathcal{P})$:

$$\mathcal{L}f := \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d a_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^d \mu_i \frac{\partial f}{\partial x_i}.$$

A $\mathcal{P}$-valued continuous adapted process $Z = (Z(t), t \ge 0)$ is called a *solution to the submartingale problem associated with* $(\mathcal{P}, R, \mu, A)$, *starting from* $x \in \mathcal{P}$, if:

(i) $Z(0) = x$ a.s.;

(ii) for every function $f \in C^2_c(\mathcal{P})$ which satisfies

$$v_i \cdot \nabla f(x) \ge 0 \ \text{ for } \ x \in \mathcal{P}_i, \ \text{ for each } \ i = 1, \ldots, m,$$

the following process is an $(\mathcal{F}_t)_{t\ge 0}$-submartingale:

$$\mathcal{M}^f = (\mathcal{M}^f(t), t \ge 0), \quad \mathcal{M}^f(t) = f(Z(t)) - \int_0^t \mathcal{L}f(Z(s))\mathrm{d}s.$$

**Lemma 5.3.5.** *The process* $\mathrm{SRBM}^d(\mathcal{P}, R, \mu, A)$*, starting from* $x \in \mathcal{P}$*, is a solution to the submartingale problem associated with* $(\mathcal{P}, R, \mu, A)$*, starting from* $x$*.*

The proof is postponed until the Appendix (Section5.6).

*5.3.5 Connection between an SRBM in the orthant and an SRBM in a convex polyhedron*

Using the linear transformation (5.12), we can switch from an $\mathrm{SRBM}^d(R, \mu, A)$ in the orthant with covariance matrix $A$ to an $\mathrm{SRBM}^d$ in a convex polyhedron with identity covariance matrix.

**Lemma 5.3.6.** *Consider the process $Z = (Z(t), t \ge 0)$, which is an $\mathrm{SRBM}^d(R, \mu, A)$. Define a new process $\overline{Z} = (\overline{Z}(t), t \ge 0)$ as follows:*

$$\overline{Z}(t) = A^{-1/2} Z(t). \tag{5.12}$$

*(i) The process $\overline{Z}$ is an $\mathrm{SRBM}^d(\mathcal{P}, \overline{R}, \overline{\mu}, I_d)$ in the convex polyhedron*

$$\mathcal{P} := \{A^{-1/2} z \mid z \in S\} = \{\overline{z} \in \mathbb{R}^d \mid A^{1/2}\overline{z} \ge 0\}, \tag{5.13}$$

*with reflection matrix $\overline{R} := A^{-1/2}R$, drift vector $\overline{\mu} := A^{-1/2}\mu$ and covariance matrix $\overline{A} = I_d$. The domain $\mathcal{P}$ is a convex polyhedron as in* (5.9) *with $m = d$ edges: $\mathcal{P}_i := \{A^{-1/2}x \mid x \in S_i\}$, $i = 1, \dots, d$. This domain satisfies the condition* (5.10) *and the Assumption* 1 *(i).*

*(ii) The standard normalization from Remark* 17 *gives us a new reflection matrix: $\tilde{R} := \overline{R}D^{1/2} = A^{-1/2}RD^{1/2}$. The $i$th column of $\tilde{R}$ is equal to*

$$v_i := a_{ii}^{1/2} A^{-1/2} R e_i, \quad i = 1, \dots, d. \tag{5.14}$$

*The inward unit normal vector to the face $\mathcal{P}_i$ is given by*

$$n_i = a_{ii}^{-1/2} A^{1/2} e_i, \quad i = 1, \dots, d. \tag{5.15}$$

*Furthermore, Assumption* 1*(ii) and (iii) is satisfied.*

*Proof.* (i) We have: $Z(t) = B(t) + RL(t)$, where $B = (B(t), t \ge 0)$ is the driving Brownian motion for the process $Z$, and $L = (L(t), t \ge 0)$ is the vector of regulating processes. Here, $B$ is a $d$-dimensional Brownian motion with drift vector $\mu$ and covariance matrix $A$. Define $W = (W(t), t \ge 0)$ as $W(t) = A^{-1/2}B(t)$: this is a $d$-dimensional Brownian motion with drift vector $\overline{\mu} = A^{-1/2}\mu$ and identity covariance matrix. Then $\overline{Z}(t) := A^{-1/2}Z(t) = W(t) + A^{-1/2}RL(t)$. The state space of $\overline{Z}$ is the domain $\mathcal{P}$, given in (5.13). This is a convex polyhedron of the type (5.9). Let us show it satisfies the condition (5.10) and the Assumption 1 (i). The linear transformation (5.12) is a bijection $\mathbb{R}^d \to \mathbb{R}^d$, hence it suffices to show that the orthant $S$ satisfies the condition (5.10) and the Assumption 1 (i), which is straightforward.

(ii) The face $\mathcal{P}_i$ is spanned by vectors $A^{-1/2}e_j,\ j \in \{1, \dots, d\} \setminus \{i\}$. The vector $n_i$ is normal to $\mathcal{P}_i$, so we must have: $n_i \cdot A^{-1/2}e_j = 0$. Since the matrix $A^{-1/2}$ is symmetric, $A^{-1/2}n_i \cdot e_j = 0$ for $j \in \{1, \dots, d\} \setminus \{i\}$. Therefore, $A^{-1/2}n_i = k_ie_i$ for some $k_i \in \mathbb{R}$; so $n_i = k_iA^{1/2}e_i$. Let us find $k_i$ such that $n_i$ is inward oriented and has unit length.

The inward orientation means that for any point $w$ in the relative interior of the face $\mathcal{P}_i$, that is, in $\mathcal{P}_i \setminus (\cup_{j\neq i}\mathcal{P}_j)$, there exists $\varepsilon > 0$ such that $w + \varepsilon n_i \in \mathcal{P}$. But the domain $\mathcal{P}$ is obtained from the orthant $S = \mathbb{R}^d_+$ by the linear transformation (5.12). So we have: $w = A^{-1/2}z$ for some $z$ in the relative interior $S_i \setminus (\cup_{j\neq i}S_j)$ of the face $S_i$ of $\partial S$. We must have $w + \varepsilon n_i \in \mathcal{P}$. But

$$w + \varepsilon n_i = A^{-1/2}\left(z + \varepsilon k_iAe_i\right), \quad \text{and} \quad \mathcal{P} = \{A^{-1/2}x \mid x \in S\}.$$

Therefore, $w + \varepsilon n_i \in \mathcal{P} \;\Leftrightarrow\; z + \varepsilon k_iAe_i \in S$. Since $z \in S_i$, we have: $z_i = 0$, and $(Ae_i)_i = a_{ii} > 0$. But $z_i + \varepsilon k_i(Ae_i)_i = (z + \varepsilon k_iAe_i)_i \geq 0$, so we must have: $k_i \geq 0$. Now, let us find $|k_i|$ using the fact that $\|n_i\| = 1$. Since the matrix $A^{1/2}$ is symmetric, we have:

$$\|A^{1/2}e_i\| = \left[A^{1/2}e_i \cdot A^{1/2}e_i\right]^{1/2} = \left[A^{1/2}(A^{1/2}e_i) \cdot e_i\right]^{1/2} = [Ae_i \cdot e_i]^{1/2} = a_{ii}^{1/2}.$$

But $\|n_i\| = 1$, and $n_i = k_iA^{1/2}e_i$. So $|k_i|a_{ii}^{1/2} = 1$, and $|k_i| = a_{ii}^{-1/2}$. Earlier, we proved that $k_i \geq 0$. Therefore, $k_i = a_{ii}^{-1/2}$, which proves (5.15). Now, let us show (5.14). The $i$th column of $A^{-1/2}R$ is equal to $A^{-1/2}Re_i$. Using the fact that the matrix $A^{1/2}$ is symmetric, we have:

$$\begin{aligned} A^{-1/2}Re_i \cdot n_i &= A^{-1/2}Re_i \cdot a_{ii}^{-1/2}A^{1/2}e_i = a_{ii}^{-1/2}A^{1/2}A^{-1/2}Re_i \cdot e_i \\ &= a_{ii}^{-1/2}Re_i \cdot e_i = a_{ii}^{-1/2}r_{ii} = a_{ii}^{-1/2}. \end{aligned}$$

Therefore, the standard normalization defined in Remark 17 leads to

$$v_i := a_{ii}^{1/2} A^{-1/2} R e_i, \quad i = 1, \ldots, d,$$

which proves (5.14). Now, let us show that the Assumption 1(ii) and (iii) is satisfied. Note that the matrix $A^{1/2}$ is symmetric, so for every $i, j = 1, \ldots, d$ we have:

$$\begin{aligned} v_i \cdot n_j =& a_{ii}^{1/2} a_{jj}^{-1/2} A^{-1/2} R e_i \cdot A^{1/2} e_j = a_{ii}^{1/2} a_{jj}^{-1/2} A^{1/2} A^{-1/2} R e_i \cdot e_j \\ &= a_{ii}^{1/2} a_{jj}^{-1/2} R e_i \cdot e_j = a_{ii}^{1/2} a_{jj}^{-1/2} r_{ij}. \end{aligned}$$

Fix a nonempty subset $I \subseteq \{1, \ldots, d\}$ with $|I| = p$. Since the matrix $R$ is completely-$\mathcal{S}$, the submatrix $[R]_I$ is an $\mathcal{S}$-matrix. There exist positive numbers $\alpha_i$, $i \in I$, such that $\sum_{j \in I} r_{ij} \alpha_j > 0$ for $i \in I$. Take $n = \sum_{j \in I} a_{jj}^{1/2} \alpha_j n_j$. This is a positive linear combination of $n_j$, $j \in I$, and $v_i \cdot n = \sum_{j \in I} a_{ii}^{1/2} r_{ij} \alpha_j > 0$ for $i \in I$. This proves Assumption 1(iii). Similarly, the transposed matrix $R'$ is also completely-$\mathcal{S}$ (this follows from Lemma 2.2.1(ii)), so repeating this argument with $R'$ in place of $R$, we can prove Assumption 1(ii). $\square$

### *5.3.6 A skew-symmetry condition for a convex polyhedron*

Consider a reflected Brownian motion in a general convex polyhedron in general dimension $d \geq 2$. Then a sufficient condition for a.s. not hitting non-smooth parts of the boundary is given by [124, Theorem 1.1]. It is called the *skew-symmetry condition*. In the subsequent exposition, we define this condition in (5.16), and show that it is equivalent (under the linear transformation (5.12)) to the skew-symmetry condition (5.5). This is the reason why these two conditions have the same name.

**Definition 27.** Consider an $\mathrm{SRBM}^d(\mathcal{P}, R, \mu, A)$ with $\mu = 0$ and $A = I_d$. Suppose the matrix $R$ is normalized, as described in Remark 17. We say that the *skew-symmetry condition* holds if

$$n_i \cdot q_j + n_j \cdot q_i = 0, \quad 1 \leq i, j \leq m. \tag{5.16}$$

This justifies the name of this condition: the matrix $(n_i \cdot q_j)_{1\le i,j\le m}$ must be skew-symmetric.

We say that an SRBM $Z = (Z(t), t \ge 0)$ *hits non-smooth parts of the boundary* $\partial\mathcal{P}$ at time $t > 0$ if there exist $1 \le i < j \le m$ such that $Z(t) \in \mathcal{P}_i \cap \mathcal{P}_j$. This is a generalization of the concept of an SRBM in the orthant hitting non-smooth parts of the boundary. For an SRBM in a two-dimensional wedge, this is equivalent to hitting the corner of the wedge (the origin): a process $Z = (Z(t), t \ge 0)$ with values in this wedge *hits the corner* at time $t > 0$ if $Z(t) = 0$.

**Proposition 5.3.7.** *Under Assumption* 1 *and the skew-symmetry condition* (5.16)*, an*

$$\mathrm{SRBM}^d(\mathcal{P}, R, \mu, A),$$

*starting from some point* $x \in \mathcal{P} \setminus \partial\mathcal{P}$ *in the interior of the polyhedral domain* $\mathcal{P}$ *a.s. does not hit non-smooth parts of the boundary at any time* $t > 0$*.*

*Proof.* Follows from Lemma 5.3.5, Proposition 5.3.4 and [124, Theorem 1.1]. □

The following lemma shows the equivalence of the two forms (5.5) and (5.16) of the skew-symmetry condition under the linear transformation (5.12).

**Lemma 5.3.8.** *Consider the process* $Z = (Z(t), t \ge 0) = \mathrm{SRBM}^d(R, \mu, A)$*. Let* $\overline{Z}$ *be the process defined by* (5.12)*. Then the skew-symmetry condition in the form* (5.5) *holds for* $Z$ *if and only if the skew-symmetry condition in the form* (5.16) *holds for* $\overline{Z}$*.*

*Proof.* Suppose (5.5) is true. Using (5.14), (5.15) and the fact that $v_i = n_i + q_i,\ i = 1, \dots, m$ (in this case $m = d$), we have:

$$\begin{aligned} n_i \cdot q_j + n_j \cdot q_i &= n_i \cdot (v_j - n_j) + n_j \cdot (v_i - n_i) = n_i \cdot v_j - n_j \cdot v_i - 2n_i \cdot n_j \\ &= a_{ii}^{-1/2} A^{1/2} e_i \cdot a_{jj}^{1/2} A^{-1/2} R e_j + a_{jj}^{-1/2} A^{1/2} e_j \cdot a_{ii}^{1/2} A^{-1/2} R e_i - 2 a_{ii}^{-1/2} a_{jj}^{-1/2} A^{1/2} e_i \cdot A^{1/2} e_j. \end{aligned}$$

Since the matrix $A^{1/2}$ is symmetric, we have:

$$a_{ii}^{-1/2} A^{1/2} e_i \cdot a_{jj}^{1/2} A^{-1/2} R e_j = a_{ii}^{-1/2} a_{jj}^{1/2} \left(e_i \cdot A^{1/2} A^{-1/2} R e_j\right)$$

$$= a_{ii}^{-1/2} a_{jj}^{1/2} \left(e_i \cdot Re_j\right) = a_{ii}^{-1/2} a_{jj}^{1/2} r_{ij},$$

similarly

$$a_{jj}^{-1/2} A^{1/2} e_j \cdot a_{ii}^{1/2} A^{-1/2} Re_i = a_{jj}^{-1/2} a_{ii}^{1/2} r_{ji},$$

and finally

$$\begin{aligned} a_{ii}^{-1/2} a_{jj}^{-1/2} A^{1/2} e_i \cdot A^{1/2} e_j &= a_{ii}^{-1/2} a_{jj}^{-1/2} \left(e_i \cdot A^{1/2} A^{1/2} e_j\right) \\ &= a_{ii}^{-1/2} a_{jj}^{-1/2} \left(e_i \cdot Ae_j\right) = a_{ii}^{-1/2} a_{jj}^{-1/2} a_{ij}. \end{aligned}$$

Therefore,

$$\begin{aligned} n_i \cdot q_j + n_j \cdot q_i &= a_{ii}^{-1/2} a_{jj}^{1/2} r_{ij} + a_{jj}^{-1/2} a_{ii}^{1/2} r_{ji} - 2 a_{ii}^{-1/2} a_{jj}^{-1/2} a_{ij} \\ &= a_{ii}^{-1/2} a_{jj}^{-1/2} \left[r_{ij} a_{jj} + r_{ji} a_{ii} - 2 a_{ij}\right] = 0. \end{aligned}$$

The converse statement is proved similarly. □

### *5.3.7 An SRBM in a two-dimensional wedge*

A particular case of a polyhedral domain is a *two-dimensional wedge* (see Fig. 1), considered in [116], [121], [122], [123]:

$$\mathcal{V} := \{(r\cos\theta, r\sin\theta) \mid 0 \le r < \infty,\ \xi_2 \le \theta \le \xi_1\}.$$

Here, $\xi_2 < \xi_1 < \xi_2 + \pi$. Its *angle* is defined as $\xi := \xi_1 - \xi_2$. Its boundary $\partial\mathcal{V}$ consists of two edges

$$\mathcal{V}_i := \{(r\cos\xi_i, r\sin\xi_i) \mid 0 \le r < \infty\}, \quad i = 1, 2.$$

The edge $\mathcal{V}_1$ is called the *upper edge*, and the edge $\mathcal{V}_2$ is called the *lower edge*. The difference between them is as follows: the shorter way to rotate $\mathcal{V}_1$ to get $\mathcal{V}_2$ is clockwise rather than counterclockwise. On each edge $\mathcal{V}_i$, there is a reflection vector $v_i$, which forms the angle $\theta_i \in (-\pi/2, \pi/2)$ with the inward unit normal vector $n_i$.

These angles are signed: positive angles $\theta_1$, $\theta_2$ are measured toward the vertex of $\mathcal{V}$ (the origin). In other words, $\theta_1$ is the angle between $n_1$ and $v_1$, measured clockwise in the direction

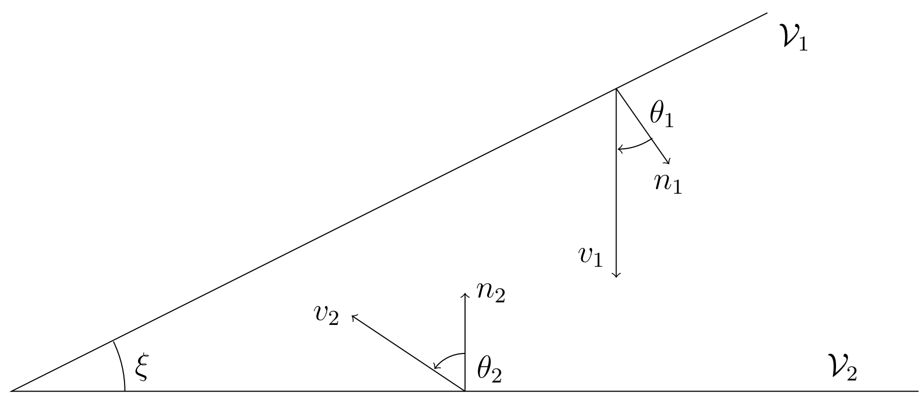


FIGURE 2. A two-dimensional wedge.
Angles $\theta_1$ and $\theta_2$ are counted toward the vertex of the wedge
Here, $n_1$ and $n_2$ are normal vectors, $v_1$ and $v_2$ are reflection vectors

from $n_1$ to $v_1$. This means the following: if the shorter way to rotate the direction of $n_1$ to get the direction of $v_1$ is clockwise, then $\theta_1 > 0$; and if it is counterclockwise, then $\theta_1 < 0$. If $v_1$ and $n_1$ have the same direction, then $\theta_1 = 0$. Simlarly, $\theta_2$ is the angle between $n_2$ and $v_2$, measured counterclockwise from $n_2$ to $v_2$.

We are interested in whether a reflected Brownian motion with zero drift vector and identity covariance matrix in this wedge hits the corner. A necessary and sufficient condition is established in [116, Theorem 2.2].

**Proposition 5.3.9.** *Consider an SRBM $Z = (Z(t), t \geq 0)$ in the wedge $\mathcal{V}$ with $\mu = 0$ and $A = I_2$, starting from a point $x \in \mathcal{V} \setminus \partial\mathcal{V}$.*

*(i) If $\theta_1 + \theta_2 > 0$, then a.s. there exists $t > 0$ such that $Z(t) = 0$.*

*(ii) If $\theta_1 + \theta_2 \leq 0$, then a.s. there does not exist $t > 0$ such that $Z(t) = 0$.*

*Proof.* Follows from Lemma 5.3.5, Proposition 5.3.4, and Theorem 2.2 from [116]. □

In the case of two dimensions, $d = 2$, the linear transformation (5.12) leads to an SRBM in a two-dimensional wedge with identity covariance matrix. In the following lemma, we explicitly calculate the parameters of this SRBM: the angle $\xi$ of this wedge and the two angles $\theta_1$, $\theta_2$ of reflection.

**Lemma 5.3.10.** *Suppose $Z = \mathrm{SRBM}^2(R, 0, A)$ and $\overline{Z}$ is the process defined by* (5.12). *Then the polyhedral domain $\mathcal{P}$ is in fact a wedge $\mathcal{V}$ with the angle*

$$\xi = \arccos\left[-\frac{a_{12}}{\sqrt{a_{11}a_{22}}}\right]. \tag{5.17}$$

*The process $\overline{Z}$ is an SRBM in $\mathcal{V}$ with zero drift vector, identity covariance matrix and the angles of reflection*

$$\theta_1 = \arcsin\frac{a_{12} - a_{11}r_{21}}{\sqrt{a_{11}\left(a_{11}r_{21}^2 - 2a_{12}r_{21} + a_{22}\right)}}, \tag{5.18}$$

$$\theta_2 = \arcsin\frac{a_{12} - a_{22}r_{12}}{\sqrt{a_{22}\left(a_{22}r_{12}^2 - 2a_{12}r_{12} + a_{11}\right)}}. \tag{5.19}$$

*Proof.* First, note that $A^{-1/2}$ is a positive definite matrix, so it has a positive determinant. Therefore, the linear transformation (5.12) preserves the orientation of the plane $\mathbb{R}^2_+$. The edges of this wedge are

$$\mathcal{V}_i := A^{-1/2}S_i \equiv \{A^{-1/2}z \mid z \in S_i\}, \quad i = 1, 2.$$

In fact, $\mathcal{V}_1$ is the upper edge, and $\mathcal{V}_2$ is the lower edge. Indeed, for the original quadrant $S = \mathbb{R}^2_+$, the edge $S_1 = \{x \in S \mid x_1 = 0\}$ is the upper edge, and the edge $S_2 = \{x \in S \mid x_2 = 0\}$ is the lower edge: in other words, the shorter way to rotate $S_1$ to get $S_2$ is clockwise rather than counterclockwise. But under the transformation 5.6, $S_1$ is mapped to $\mathcal{V}_1$, and $S_2$ is mapped to $\mathcal{V}_2$. This linear transformation preserves the orientation. Therefore, the shorter way to rotate $\mathcal{V}_1$ to get $\mathcal{V}_2$ is also clockwise rather than counterclockwise. The edge $\mathcal{V}_1$ has a directional vector $c_2 = A^{-1/2}e_2$, while the edge $\mathcal{V}_2$ has a directional vector $c_1 = A^{-1/2}e_1$. An important remark: consider the notation $\mathcal{P}_i,\ i = 1, \ldots, d$, for edges of the polyhedron from Lemma 5.3.6. Then our current notation $\mathcal{V}_1$ and $\mathcal{V}_2$ is consistent with this notation in the sense that

$$\mathcal{V}_1 = \mathcal{P}_1 \ \text{ and } \ \mathcal{V}_2 = \mathcal{P}_2. \tag{5.20}$$

The angle $\xi$ of the wedge is the angle between the edges $\mathcal{V}_1$ and $\mathcal{V}_2$. So $\xi$ is the angle between two vectors $c_1 = A^{-1/2}e_1$ and $c_2 = A^{-1/2}e_2$. Since the matrix $A^{-1/2}$ is symmetric, we have:

$$\cos\xi = \frac{A^{-1/2}e_1 \cdot A^{-1/2}e_2}{\|A^{-1/2}e_1\|\|A^{-1/2}e_2\|} = \frac{(A^{-1/2})^2e_1 \cdot e_2}{\left[(A^{-1/2})^2e_1 \cdot e_1\right]^{1/2}\left[(A^{-1/2})^2e_2 \cdot e_2\right]^{1/2}}$$

$$= \frac{A^{-1}e_1 \cdot e_2}{\left[A^{-1}e_1 \cdot e_1\right]^{1/2}\left[A^{-1}e_2 \cdot e_2\right]^{1/2}} = \frac{(A^{-1})_{12}}{(A^{-1})_{11}^{1/2}(A^{-1})_{22}^{1/2}}.$$

But

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}^2}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{12} & a_{11} \end{bmatrix} \tag{5.21}$$

Therefore,

$$\cos\xi = -\frac{a_{12}}{\sqrt{a_{11}a_{22}}},$$

and we get (5.17). Let us find the reflection angles $\theta_1$ and $\theta_2$. For the quadrant $S = \mathbb{R}_+^2$, if we rotate the directional vector $e_2$ of the upper face $S_1$ clockwise by $\pi/2$, we get an inward normal vector to this face. But the linear transformation (5.12) preserves the orientation, so a similar statement is true for the wedge $\mathcal{V}$: if we rotate the directional vector $c_2 = A^{-1/2}e_2$ of the upper face $\mathcal{V}_1$ of the wedge clockwise by $\pi/2$, then we get an inward normal vector

$$\mathfrak{n}_1 \equiv \begin{bmatrix} (\mathfrak{n}_1)_1 \\ (\mathfrak{n}_1)_2 \end{bmatrix} := \begin{bmatrix} (c_2)_2 \\ -(c_2)_1 \end{bmatrix}$$

Similarly, if we rotate the vector $c_1 = A^{-1/2}e_1$ by $\pi/2$ counterclockwise, we get an inward normal vector

$$\mathfrak{n}_2 \equiv \begin{bmatrix} (\mathfrak{n}_2)_1 \\ (\mathfrak{n}_2)_2 \end{bmatrix} := \begin{bmatrix} -(c_1)_2 \\ (c_1)_1 \end{bmatrix}$$

to $\mathcal{V}_1$. These are not unit vectors: $\mathfrak{n}_i \neq n_i$. In fact, $\|\mathfrak{n}_1\| = \|c_2\|$ and $\|\mathfrak{n}_2\| = \|c_1\|$. But $\mathfrak{n}_1$ has the same direction as $n_1$, and $\mathfrak{n}_2$ has the same direction as $n_2$. In other words, $\mathfrak{n}_1 = \|\mathfrak{n}_1\|n_1$ and $\mathfrak{n}_2 = \|\mathfrak{n}_2\|n_2$.

From Lemma 5.3.6 and (5.20), it follows that $v_1 = A^{-1/2}r_1$ and $v_2 = A^{-1/2}r_2$. These vectors are *not normalized* in the sense of Remark 17. The angle $\theta_1$ between $n_1$ and $v_1$ has a sign: it is calculated toward the origin, or, in other words, counterclockwise from $n_1$ to $v_1$. But $n_1$ and $\mathfrak{n}_1$ have the same direction. Therefore, $\theta_1$ can be calculated as the signed angle from $\mathfrak{n}_1$ to $v_1$ in the counterclockwise direction:

$$\sin\theta_1 = \frac{(\mathfrak{n}_1)_1(v_1)_2 - (\mathfrak{n}_1)_2(v_1)_1}{\|\mathfrak{n}_1\|\|v_1\|} = \frac{-(c_2)_2(v_1)_2 - (c_2)_1(v_1)_1}{\|c_2\|\|v_1\|} = -\frac{c_2 \cdot v_1}{\|c_2\|\|v_1\|}$$

$$= -\frac{A^{-1/2}e_2 \cdot A^{-1/2}r_1}{\|A^{-1/2}e_2\|\|A^{-1/2}r_1\|} = -\frac{A^{-1/2}e_2 \cdot A^{-1/2}r_1}{\left[A^{-1/2}e_2 \cdot A^{-1/2}r_1\right]^{1/2}\left[A^{-1/2}e_2 \cdot A^{-1/2}r_1\right]^{1/2}}$$

Since the matrix $A^{-1/2}$ is symmetric, the last expression is equal to

$$-\frac{A^{-1}e_2 \cdot r_1}{\left[A^{-1}e_2 \cdot e_2\right]^{1/2}\left[A^{-1}r_1 \cdot r_1\right]^{1/2}}.$$

Using the formula (5.21) for $A^{-1}$ and the fact that $r_1 = (1, r_{21})'$, we have:

$$\sin\theta_1 = \frac{a_{12} - a_{11}r_{21}}{\sqrt{a_{11}\left(a_{11}r_{21}^2 - 2a_{12}r_{21} + a_{22}\right)}}.$$

Similarly, we can calculate the angle $\theta_2$:

$$\sin\theta_2 = \frac{a_{12} - a_{22}r_{12}}{\sqrt{a_{22}\left(a_{22}r_{12}^2 - 2a_{12}r_{12} + a_{11}\right)}}.$$

Since $\theta_1, \theta_2 \in (-\pi/2, \pi/2)$, we get (5.18) and (5.19). □

### *5.3.8 Completion of the proof of Theorem* 5.2.1

By Lemma 5.3.2, without loss of generality we can assume an SRBM starts from some point $x \in S \setminus \partial S$, and $\mu = 0$. First, we prove (i) in the case of the skew-symmetry condition (5.5), then move to the general case (5.4). Then we prove (ii) in the case $d = 2$, and proceed to the case of the general dimension.

**Lemma 5.3.11.** *Take an SRBM in the orthant $S$, starting from $x \in S \setminus \partial S$. Suppose it satisfies the skew-symmetry condition* (5.5). *Then the statement of Theorem* 5.2.1*(i) is true.*

*Proof.* Apply the linear transformation (5.12) to $Z = (Z(t), t \geq 0) = \mathrm{SRBM}^d(R, 0, A)$. By Lemma 5.3.6, we get an SRBM $\overline{Z} = (\overline{Z}(t), t \geq 0)$ in the polyhedron $\mathcal{S} = A^{-1/2}S$, given by (5.13) with zero drift and identity covariance matrix. It was shown in Lemma 5.3.8 that the skew-symmetry condition (5.16) is true. Therefore, by Proposition 5.3.7 the process $\overline{Z}$ a.s. does not hit non-smooth parts of the boundary $\partial\mathcal{S}$ at any moment $t > 0$. Thus, the process $Z$ a.s. does not hit non-smooth parts of the boundary $\partial S$ at any moment $t > 0$. □

**Lemma 5.3.12.** *Take an SRBM in the orthant $S$, starting from $x \in S \setminus \partial S$. Suppose it satisfies the condition* (5.4)*. Then the statement of Theorem* 5.2.1*(i) is true.*

*Proof.* Let us find another reflection nonsingular $\mathcal{M}$-matrix $\tilde{R} = (\tilde{r}_{ij})_{1\le i,j\le d}$ such that $R \ge \tilde{R}$, and the skew-symmetry condition (5.5) is true for an $\mathrm{SRBM}^d(\tilde{R}, 0, A)$. We need:

$$\tilde{r}_{ij}a_{jj} + \tilde{r}_{ji}a_{ii} = 2a_{ij},\ i, j = 1, \ldots, d. \tag{5.22}$$

Let $\tilde{r}_{ij} = 1$ for $i = j$. Then (5.22) is true for $i = j$. Let

$$\tilde{r}_{ij} = \frac{1}{a_{jj}}\left[2a_{ij} - r_{ji}a_{ii}\right],\ \tilde{r}_{ji} = r_{ji},\quad 1 \le i < j \le d.$$

This is well defined, since $a_{jj} > 0$ (because the matrix $A$ is positive definite). Also, $\tilde{r}_{ij} \le r_{ij}$, because $r_{ij}a_{jj} + r_{ji}a_{ii} \ge 2a_{ij}$. Since $\tilde{r}_{ij} \le r_{ij} \le 0$ for $i \ne j$, $\tilde{R}$ is a $\mathcal{Z}$-matrix, so condition (5.22) holds. Therefore, by [55, Theorem 2.5] (compare conditions 12 and 16), $\tilde{R}$ is a nonsingular $\mathcal{M}$-matrix. Consider two processes $Z = \mathrm{SRBM}^d(R, \mu, A)$, $\tilde{Z} = \mathrm{SRBM}^d(\tilde{R}, \mu, A)$, starting from the same initial condition $x \in S \setminus \partial S$. Then we have: $R$ and $\tilde{R}$ are $d \times d$ reflection nonsingular $\mathcal{M}$-matrices, and $R \ge \tilde{R}$. By Proposition 4.3.5, we have: $\tilde{Z}$ is stochastically smaller than $Z$. By [70, Theorem 5], we can claim that a.s. for all $t > 0$ we have: $\tilde{Z}(t) \le Z(t)$ (possibly after changing the probability space). By Lemma 5.3.11, the process $\tilde{Z}$ a.s. does not hit non-smooth parts of the boundary at any time $t > 0$. In other words, for every $1 \le i < j \le d$, we have: a.s. $\tilde{Z}_i(t) + \tilde{Z}_j(t) > 0$ for all $t > 0$. Therefore, a.s. $Z_i(t) + Z_j(t) > 0$ for all $t > 0$. Thus, with probability one the process $Z$ does not hit non-smooth parts of the boundary at any time $t > 0$. $\square$

Now, let us prove part (ii) of Theorem 5.2.1. We start with the case $d = 2$, then move to the general case.

**Lemma 5.3.13.** *Suppose we start an SRBM in two dimensions from a point $x \in S \setminus \partial S$ in the interior of $S$. Then the statement of Theorem* 5.2.1 *(ii) is valid.*

*Proof.* Let $Z = (Z(t), t \ge 0) = \mathrm{SRBM}^2(R, 0, A)$. After the linear transformation (5.12), we get the process $\overline{Z} = (\overline{Z}(t), t \ge 0)$ from (5.12), which is an SRBM in a wedge. If we show that $\theta_1 + \theta_2 > 0$, then by Lemma 5.3.9 we have: a.s. there exists $t > 0$ such that $\overline{Z}(t) \equiv A^{-1/2}Z(t) = 0$; therefore, a.s. there exists $t > 0$ such that $Z(t) = 0$. But the angles $\theta_1$, $\theta_2$ are given in the equations (5.18) and (5.19). Since $\theta_1, \theta_2 \in (-\pi/2, \pi/2)$, we have:

$$\theta_1 + \theta_2 > 0 \quad \Leftrightarrow \quad \sin\theta_1 + \sin\theta_2 > 0,$$

which can be written as

$$\frac{a_{11}r_{21} - a_{12}}{\sqrt{a_{11}\left(a_{11}r_{21}^2 - 2a_{12}r_{21} + a_{22}\right)}} + \frac{a_{22}r_{12} - a_{12}}{\sqrt{a_{22}\left(a_{22}r_{12}^2 - 2a_{12}r_{12} + a_{11}\right)}} < 0. \tag{5.23}$$

Then we have:

$$r_{12}' := a_{11}^{-1/2}a_{22}^{1/2}r_{12}, \quad r_{21}' = a_{11}^{1/2}a_{22}^{-1/2}r_{21}, \quad \rho := a_{11}^{-1/2}a_{22}^{-1/2}a_{12}.$$

We can rewrite the condition (5.23) as

$$\frac{r_{12}' - \rho}{\sqrt{(r_{12}')^2 - 2\rho r_{12}' + 1}} + \frac{r_{21}' - \rho}{\sqrt{(r_{21}')^2 - 2\rho r_{21}' + 1}} < 0.$$

Or, equivalently, $f(r_{12}' - \rho) + f(r_{21}' - \rho) < 0$, where

$$f(x) := \frac{x}{\sqrt{x^2 + 1 - \rho^2}}.$$

Note that the matrix $A$ is positive definite, so $\det A = a_{11}a_{22} - a_{12}^2 > 0$. Therefore, $\rho^2 < 1$. It is easy to show that the function $f$ is strictly increasing on $\mathbb{R}$. In addition, this function is odd: $f(x) + f(-x) \equiv 0$. Therefore, $f(r_{12}' - \rho) + f(r_{21}' - \rho) < 0$ is equivalent to

$$(r_{12}' - \rho) + (r_{21}' - \rho) < 0 \quad \Leftrightarrow \quad r_{12}a_{22} + r_{21}a_{11} < 2a_{12}.$$

$\square$

**Lemma 5.3.14.** *The statement (ii) of Theorem* 5.2.1 *is valid in the case of general dimension, if we start an SRBM from a point $x \in S \setminus \partial S$ in the interior of $S$.*

*Proof.* Let $Z = \mathrm{SRBM}^d(R, 0, A)$. Assume now that the condition (5.4) is not true, and for some $1 \le i < j \le d$ we have:

$$r_{ij}a_{jj} + r_{ji}a_{ii} < 2a_{ij}. \tag{5.24}$$

Consider the following two-dimensional SRBM: $\tilde{Z} = \mathrm{SRBM}^2([R]_I, 0, [A]_I)$, where $I = \{i, j\}$. Applying Corollary 4.3.4 from Chapter 4 to $I := \{i, j\}$, we get: $[Z]_I \preceq \tilde{Z}$. By [70, Theorem 5], we can switch from stochastic comparison to pathwise comparison: after changing the probability space, we can claim that a.s. for all $t > 0$ we have: $[Z(t)]_I \le \tilde{Z}(t)$. By Lemma 5.3.13, with positive probability, there exists $t > 0$ such that $\tilde{Z}_i(t) = \tilde{Z}_j(t) = 0$. Therefore, with positive probability there exists $t > 0$ such that $Z_i(t) = Z_j(t) = 0$. $\square$

## 5.4 *Proof of Theorems* 5.1.1 *and* 5.1.3

Theorem 5.1.3 can be easily deduced from Theorem 5.2.1. First, let us prove part (i) of Theorem 5.1.3. We need to rewrite the condition (5.4) for concrete matrices $R$ and $A$ arising from competing Brownian particles, given by (3.6) and (3.8). Take $i, j = 1, \ldots, N-1$ and consider the condition

$$r_{ij}a_{jj} + r_{ji}a_{ii} \ge 2a_{ij}. \tag{5.25}$$

If $i = j$, then (5.25) is always true, because for such $i$, $j$ we have: $r_{ij} = r_{ji} = 1$, and $a_{ii} = a_{ij} = a_{jj} = \sigma_i^2 + \sigma_{i+1}^2$. If $|i - j| \ge 2$, then (5.25) is also always true, since $r_{ij} = r_{ji} = a_{ij} = 0$. Since the left-hand side and the right-hand side of (5.25) remain the same if we swap $i$ and $j$, we need only to check this condition for $j = k$, $i = k-1$, where $k = 2, \ldots, N-1$. We get:

$$r_{ij} = -q_k^-,\quad r_{ji} = -q_k^+,\ a_{jj} = \sigma_k^2 + \sigma_{k+1}^2,\ a_{ii} = \sigma_{k-1}^2 + \sigma_k^2,\ a_{ij} = -\sigma_k^2.$$

Therefore, the condition (5.25) takes the form

$$-q_k^-\left(\sigma_k^2 + \sigma_{k+1}^2\right) - q_k^+\left(\sigma_{k-1}^2 + \sigma_k^2\right) \ge -2\sigma_k^2.$$

This is equivalent to

$$\left(2 - q_k^- - q_k^+\right)\sigma_k^2 \ge q_k^-\sigma_{k+1}^2 + q_k^+\sigma_{k-1}^2. \tag{5.26}$$

Note that $q_k^- + q_{k+1}^+ = 1$ and $q_k^+ + q_{k-1}^- = 1$. Therefore, we can rewrite (5.26) as in (5.3). This proves part (i) of Theorem 5.1.3. Now, let us prove part (ii) of this theorem. Since the condition (5.4) is automatically valid for $i = j$ and for $|i - j| \ge 2$, it can be violated only for

$i = j - 1$. Suppose it does not hold for $j = k$ and $i = k - 1$, where $k = 2, \ldots, N - 1$ is some index. Then with positive probability, there exists $t > 0$ such that

$$Z_{k-1}(t) = Z_k(t) = 0,$$

which can be written as

$$Y_{k-1}(t) = Y_k(t) = Y_{k+1}(t).$$

This means that with positive probability, there is a triple collision between particles with ranks $k - 1$, $k$ and $k + 1$. This completes the proof of Theorem 5.1.3.

Theorem 5.1.1 is simply a corollary of Theorem 5.1.3: just plug parameters of collision $q_k^{\pm} = 1/2$, $k = 1, \ldots, N$ into the inequality (5.3).

*Remark* 18. Let us explain the meaning of Corollary 5.1.2 informally. Consider the gap process of a system of competing Brownian particles from Definition 14. This is an SRBM $Z = (Z(t), t \ge 0)$ in the orthant with reflection matrix $R$ and covariance matrix $A$, given by (3.6) and (3.8). In this case, the condition (5.4) can be violated only for $i = j-1$, because for $i = j$ and $|i - j| \ge 2$ it is automatically true.

When $Z_i(t) = Z_j(t) = 0$ for $1 \le i < j \le d$, this corresponds to a simultaneous collision at time $t$ in this system of competing Brownian particles: $Y_i(t) = Y_{i+1}(t)$ and $Y_j(t) = Y_{j+1}(t)$. But if, in addition, we know that $i = j - 1$, then this is a particular case of a simultaneous collision: namely, a *triple collision* between particles with ranks $j - 1$, $j$ and $j + 1$. This implies that if the condition (5.4) does not hold, then with positive probability there occurs a simultaneous collision of a special kind: a triple collision. This is the reason why Corollary 5.1.2 is true.

## 5.5 *Appendix: Proof of Lemma* 5.3.5

Recall that the process $Z = (Z(t), t \ge 0)$ which is an $\mathrm{SRBM}^d(\mathcal{P}, R, \mu, A)$ can be represented as $Z(s) = B(t) + RL(t)$. Here, $B = (B(t), t \ge 0)$ is a $d$-dimensional Brownian motion with drift vector $\mu$ and covariance matrix $A = (a_{ij})_{1\le i,j\le d}$; $R = (r_{ij})$ is an $m \times d$-matrix, and

$L = (L_1, \ldots, L_m)'$, where each $L_i$ is nondecreasing. Therefore, the mutual variation of the components of $Z$ is calculated as follows: $\langle Z_i, Z_j \rangle_t = a_{ij} t$, for $i, j = 1, \ldots, d$. The process $(B_i(s) - \mu_i s, s \geq 0)$ is a one-dimensional driftless Brownian motion. Since $f \in C_c^2(\mathcal{P})$, the following process is a martingale:

$$M(t) = \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(Z(s)) \mathrm{d}(B_i(s) - \mu_i s).$$

Apply the Itô-Tanaka formula to $f(Z(t))$:

$$\begin{aligned}
f(Z(t)) - f(Z(0)) = & \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(Z(s)) \mathrm{d}Z(s) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(Z(s)) \mathrm{d}\langle Z_i, Z_j \rangle_s \\
= & \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(Z(s)) \mathrm{d}\left(B_i(s) - \mu_i s\right) + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(Z(s)) \mu_i \mathrm{d}s \\
& + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(Z(s)) \mathrm{d}s + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(Z(s)) \mathrm{d}\left[\sum_{j=1}^{m} r_{ij} L_j(s)\right] \\
= & \, M(t) + \int_0^t \mathcal{L} f(Z(s)) \mathrm{d}s + \sum_{i=1}^{d} \sum_{j=1}^{m} \int_0^t r_{ij} \frac{\partial f}{\partial x_i}(Z(s)) \mathrm{d}L_j(s) \\
= & \, M(t) + \int_0^t \mathcal{L} f(Z(s)) \mathrm{d}s + \sum_{j=1}^{m} \int_0^t v_j \cdot \nabla f(Z(s)) \mathrm{d}L_j(s).
\end{aligned}$$

The third term in the last sum is nondecreasing. Indeed, for each $j = 1, \ldots, m$, the process $L_j$ is nondecreasing, and it can increase only when $Z(s) \in \mathcal{P}_j$. But in this case, $v_j \cdot \nabla f(Z(s)) \geq 0$. The rest is trivial.

# Chapter 6
# MULTIPLE COLLISIONS

In this chapter, which corresponds to the author's paper [102], we formulate general theorems about an $\mathrm{SRBM}^d(R, \mu, A)$ avoiding an edge

$$S_I := \{z \in S \mid z_i = 0 \text{ for all } i \in I\}$$

of the boundary $\partial S$, where $I \subseteq \{1, \ldots, d\}$ is a nonempty subset. We also find sufficient conditions for avoiding collisions of competing Brownian particles. Examples 1.2.2, 1.2.3 and 1.2.4 from the Introduction are corollaries of the general results from this chapter.

The chapter is organized as follows. In Section 6.2, we state a few necessary definitions. In Section 6.3, we formulate main results for classical systems of competing Brownian particles. In Section 6.4, we state and prove results for an SRBM, which are used in Section 6.5 to prove theorems from Section 6.3. In Section 6.6, we prove Theorem 6.2.2 for $N = 4$, which is isolated from other results and cannot be generalized to $N \geq 5$ (but this result is not weaker than the other results). In Section 6.7, we consider the case of asymmetric collisions. Although we do not state explicitly results for systems of competing Brownian particles with asymmetric collisions, they can be derived from the general statements of Section 6.4. In Section 6.8 (Appendix), we state and prove some technical lemmas.

## *6.1 Definitions*

**Definition 28.** Consider a classical system of competing Brownian particles from Definition 12. We say that a *collision of order $M$ occurs at time* $t \geq 0$, if there exists $k = 1, \ldots, N$ such that

$$Y_k(t) = Y_{k+1}(t) = \ldots = Y_{k+M}(t).$$

A collision of order $M = 2$ is called a *triple collision*. A collision of order $M = N - 1$ is called a *total collision*.

As mentioned before, a related example of a total collision (for a slightly different SDE) was considered in the paper [6].

There is another closely related concept. We can have, for example, $Y_1(t) = Y_2(t)$ and $Y_4(t) = Y_5(t) = Y_6(t)$ at the same moment $t \ge 0$. This is called a *multicollision* of a certain order (this particular one is of order 3).

**Definition 29.** Consider a classical system of competing Brownian particles from Definition 12, and fix a nonempty subset $I \subseteq \{1, \ldots, N-1\}$. A *multicollision with pattern* $I$ occurs at time $t \ge 0$ if

$$Y_k(t) = Y_{k+1}(t), \quad \text{for all} \quad k \in I.$$

We shall sometimes say that *there are no multicollisions with pattern* $I$ if a.s. there does not exist $t > 0$ such that there is a multicollision with pattern $I$ at time $t$.

A multicollision with pattern $I$ has order $M = |I|$. If $I = \{k, k+1, \ldots, l-2, l-1\}$, then a multicollision with pattern $I$ is, in fact, a multiple collision of particles with ranks $k, k+1, \ldots, l-1, l$. If $I = \{1, \ldots, N-1\}$, this is a total collision. If $I = \{k, l\}$, this is a simultaneous collision. If $I = \{k, k+1\}$, this is a triple collision.

It is worth providing some references about a diffusion hitting a lower-dimensional manifold: the articles [39], [91], [92], [10], and the book [40].

## *6.2 Results for Competing Brownian Particles: Theorems* **6.2.1, 6.2.3** *and* **6.2.2**

### *6.2.1 Sufficient conditions for avoiding total collisions*

Let us introduce some additional notation. Let $M \ge 2$. For

$$\alpha = (\alpha_1, \ldots, \alpha_M)' \in \mathbb{R}^M \quad \text{and} \quad l = 1, \ldots, M-1,$$

we define

$$c_l(\alpha) := -\frac{2(M-1)}{M}\alpha_1^2 + \frac{2(M+1)}{M}\sum_{p=2}^{l}\alpha_p^2 + \frac{2(M-1)(M-l)-4l}{(M-l)M}\sum_{p=l+1}^{M}\alpha_p^2.$$

We also denote by $\alpha^{\leftarrow} := (\alpha_M, \dots, \alpha_1)'$ the vector $\alpha$ with components put in the reverse order. Note that $c_{M-1}(\alpha) = c_{M-1}\left(\alpha^{\leftarrow}\right)$. Let

$$\mathcal{P}(\alpha) := \min\left(c_1(\alpha),\, c_1\left(\alpha^{\leftarrow}\right),\, c_2(\alpha),\, c_2\left(\alpha^{\leftarrow}\right),\, \dots,\, c_{M-2}(\alpha),\, c_{M-2}\left(\alpha^{\leftarrow}\right),\, c_{M-1}(\alpha)\right). \quad (6.1)$$

For example, in cases $M = 2$ and $M = 3$ we have the following expressions for $\mathcal{P}(\alpha)$:

$$\mathcal{P}(\alpha_1, \alpha_2) = c_1(\alpha_1, \alpha_2) = -\alpha_1^2 - \alpha_2^2\,, \quad (6.2)$$

$$\mathcal{P}(\alpha_1, \alpha_2, \alpha_3) = \min\left(\frac{8}{3}\alpha_2^2 - \frac{4}{3}\alpha_1^2 - \frac{4}{3}\alpha_3^2,\ \ \frac{2}{3}\alpha_2^2 + \frac{2}{3}\alpha_3^2 - \frac{4}{3}\alpha_1^2,\ \ \frac{2}{3}\alpha_1^2 + \frac{2}{3}\alpha_2^2 - \frac{4}{3}\alpha_3^2\right). \quad (6.3)$$

**Theorem 6.2.1.** *Consider a classical system of competing Brownian particles from Definition* 12*, and denote*

$$\sigma := (\sigma_1, \dots, \sigma_N)'.$$

*If $\mathcal{P}(\sigma) \ge 0$ in the notation of* (6.1)*, then a.s. there is no total collision at any time $t > 0$.*

By modifying the proof of Theorem 6.2.1, one can obtain other conditions for lack of total collisions. Unlike Theorem 6.2.1, however, this new result works only for $N = 4$ particles. This result is due to Cameron Bruggeman.

**Theorem 6.2.2.** *With $N = 4$ in the setting of Theorem* 6.2.1*, if*

$$\sigma_1^2 + \sigma_4^2 \le \sigma_2^2 + \sigma_3^2, \quad (6.4)$$

*then a.s. there are no total collisions at any time $t > 0$.*

As we will demonstrate in the following examples, neither set of conditions is strictly stronger than the other.

### 6.2.2 Examples of avoiding total collisions

In this subsection, we consider systems of $N = 3$, $N = 4$ and $N = 5$ particles. We apply Theorem 6.2.1 to find a sufficient condition for a.s. avoiding total collisions. In particular, we compare our results for three particles to a necessary and sufficient condition (5.1). We also compare results for $N = 4$ particles given by Theorem (6.2.1) and Theorem 6.2.2.

*Example* 2. *The case of $N = 3$ particles.* In this case, "triple collision" is a synonym for "total collision". The quantity $\mathcal{P}(\sigma)$ is calcluated in (6.3), so $\mathcal{P}(\sigma) \ge 0$ is equivalent to

$$\begin{cases} \sigma_1^2 + \sigma_3^2 \le 2\sigma_2^2; \\ 2\sigma_1^2 \le \sigma_2^2 + \sigma_3^2; \\ 2\sigma_3^2 \le \sigma_2^2 + \sigma_1^2. \end{cases} \tag{6.5}$$

In fact, the first inequality in (6.5) follows from the second and the third ones. So (6.5) is equivalent to

$$\begin{cases} 2\sigma_1^2 \le \sigma_2^2 + \sigma_3^2; \\ 2\sigma_3^2 \le \sigma_2^2 + \sigma_1^2. \end{cases} \tag{6.6}$$

This sufficient condition is more restrictive than (5.1), which for $N = 3$ particles takes the form $2\sigma_2^2 \ge \sigma_1^2 + \sigma_3^2$, so Theorem 6.2.1 gives a weaker result than the result from Chapter 5, mentioned in Proposition 5.1.1.

*Example* 3. *The case of $N = 4$ particles.* This result was already mentioned in the Introduction as Proposition 1.2.2. The condition $\mathcal{P}(\sigma) \ge 0$ holds, if and only if all the following five inequalities hold:

$$\begin{cases} 9\sigma_1^2 \le 7\sigma_2^2 + 7\sigma_3^2 + 7\sigma_4^2; \\ 3\sigma_1^2 \le 5\sigma_2^2 + \sigma_3^2 + \sigma_4^2; \\ 3\sigma_1^2 + 3\sigma_4^2 \le 5\sigma_2^2 + 5\sigma_3^2; \\ 3\sigma_4^2 \le \sigma_1^2 + \sigma_2^2 + 5\sigma_3^2; \\ 9\sigma_4^2 \le 7\sigma_1^2 + 7\sigma_2^2 + 7\sigma_3^2. \end{cases} \tag{6.7}$$

As mentioned in Section 6.1, let $\sigma_1^2 = \sigma_2^2 = \sigma_4^2 = 1$, and $\sigma_3^2 = 0.9$. Then there are triple collisions between the particles $Y_2$, $Y_3$ and $Y_4$ with positive probability, because the sequence $(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2)$ is not concave: it does not satisfy the condition (5.1). But the condition $\mathcal{P}(\sigma) \geq 0$ is satisfied, so there are a.s. no total collisions. Note that this example satisfies the conditions of Theorem 6.2.1, but fails to satisfy those of Theorem 6.2.2.

*Example* 4. *The case of* $N = 5$ *particles.* In this case $\mathcal{P}(\sigma) \geq 0$ is equivalent to the following seven inequalities:

$$\begin{cases} 8\sigma_1^2 \leq 7\sigma_2^2 + 7\sigma_3^2 + 7\sigma_4^2 + 7\sigma_5^2; \\ 6\sigma_1^2 \leq 9\sigma_2^2 + 4\sigma_3^2 + 4\sigma_4^2 + 4\sigma_5^2; \\ 4\sigma_1^2 \leq 6\sigma_2^2 + 6\sigma_3^2 + \sigma_4^2 + \sigma_5^2; \\ 2\sigma_1^2 + 2\sigma_5^2 \leq 3\sigma_2^2 + 3\sigma_3^2 + 3\sigma_4^2; \\ 8\sigma_5^2 \leq 7\sigma_4^2 + 7\sigma_3^2 + 7\sigma_2^2 + 7\sigma_1^2; \\ 6\sigma_5^2 \leq 9\sigma_4^2 + 4\sigma_3^2 + 4\sigma_2^2 + 4\sigma_1^2; \\ 4\sigma_5^2 \leq 6\sigma_4^2 + 6\sigma_3^2 + \sigma_2^2 + \sigma_1^2. \end{cases} \tag{6.8}$$

By analogy with the previous example, let $\sigma_1^2 = \sigma_2^2 = \sigma_4^2 = \sigma_5^2 = 1$, and $\sigma_3^2 = 0.9$. Then there are triple collisions among the particles $Y_2$, $Y_3$ and $Y_4$ with positive probability, but a.s. no total collisions.

*Example* 5. *An application of Theorem* 6.2.2. Take $\sigma_1^2 = \sigma_3^2 = 10$ and $\sigma_2^2 = \sigma_4^2 = 1$. Then by Theorem 6.2.2 there are a.s. no total collisions, but this fails to satisfy the conditions of Theorem 6.2.1. This, together with Example 3, shows that none of the two results: Theorem 6.2.1 applied to the case of $N = 4$ particles, and Theorem 6.2.2, is stronger than the other one.

### *6.2.3 A sufficient condition for avoiding multicollisions of a given pattern*

For every nonempty finite subset $I \subseteq \mathbb{Z}$, denote by $\overline{I} := I \cup \{\max I + 1\}$ the augmentation of $I$ by the integer following its maximal element. For example, if $I = \{1, 2, 4, 6\}$, then

$\overline{I} = \{1, 2, 4, 6, 7\}$. A nonempty finite subset $I \subseteq \mathbb{Z}$ is called a *discrete interval* if it has the form $\{k, k+1, \ldots, l-1, l\}$ for some $k, l \in \mathbb{Z},\ k \le l$. For example, the sets $\{2\}, \{3, 4\}, \{-2, -1, 0\}$ are discrete intervals, and the set $\{3, 4, 6\}$ is not. Two disjoint discrete intervals are called *adjacent* if their union is also a discrete interval. For example, discrete intervals $\{1, 2\}$ and $\{3, 4\}$ are adjacent, while $\{3, 4, 5\}$ and $\{10, 11\}$ are not.

Every nonempty finite subset $I \subseteq \mathbb{Z}$ can be decomposed into a finite union of disjoint non-adjacent discrete intervals: for example, $I = \{1, 2, 4, 8, 9, 10, 11, 13\}$ can be decomposed as $\{1, 2\} \cup \{4\} \cup \{8, 9, 10, 11\} \cup \{13\}$. This decomposition is unique. The non-adjacency is necessary for uniqueness: for example, $\{1, 2\} \cup \{4\} \cup \{8, 9, 10\} \cup \{11\} \cup \{13\}$ is also a decomposition into a finite union of disjoint discrete intervals, but $\{8, 9, 10\}$ and $\{11\}$ are adjacent.

For a vector $\alpha = (\alpha_1, \ldots, \alpha_M)' \in \mathbb{R}^M$, define

$$\mathcal{T}(\alpha) = \frac{2(M-1)}{M} \sum_{p=1}^{M} \alpha_p^2. \tag{6.9}$$

For every discrete interval $I = \{k, \ldots, l\} \subseteq \{1, \ldots, N\}$, let $\mathcal{P}(I) := \mathcal{P}(\sigma_k, \ldots, \sigma_l)$ and $\mathcal{T}(I) := \mathcal{T}(\sigma_k, \ldots, \sigma_l)$.

Consider a subset $I \subseteq \{1, \ldots, N-1\}$. Suppose it has the following decomposition into the union of non-adjacent discrete disjoint intervals:

$$I = I_1 \cup I_2 \cup \ldots \cup I_r. \tag{6.10}$$

**Definition 30.** We say that $I$ *satisfies assumption (A)* if

$$\sum_{\substack{j=1 \\ j \ne i}}^{r} \mathcal{T}(\overline{I}_j) + \mathcal{P}(\overline{I}_i) \ge 0, \quad i = 1, \ldots, r. \tag{6.11}$$

We say that $I$ *satisfies assumption (B)* if at least one of the following is true:

- at least two of discrete intervals $I_1, \ldots, I_r$ are singletons;

- at least one of discrete intervals $I_1, \dots, I_r$ consists of two elements $\{k-1, k\}$, and the sequence $(\sigma_j^2)$ has *local concavity* at $k$:

$$\sigma_k^2 \ge \frac{1}{2}\left(\sigma_{k-1}^2 + \sigma_{k+1}^2\right); \tag{6.12}$$

- there exists a subset

$$I' = I_{i_1} \cup I_{i_2} \cup \ldots \cup I_{i_s}$$

which satisfies the assumption (A).

*Remark* 19. (i) If a subset $I \subseteq \{1, \dots, N-1\}$ is a discrete interval, that is, the decomposition (6.10) is trivial, then Assumption (A) is equivalent to $\mathcal{P}(\overline{I}) \ge 0$.

(ii) If a subset $I \subseteq \{1, \dots, N-1\}$ is a discrete interval of three or more elements, then Assumption (B) is equivalent to $\mathcal{P}(\overline{I}) \ge 0$.

(iii) If a subset $I \subseteq \{1, \dots, N-1\}$ contains two elements: $I = \{k, l\},\ k < l$, then Assumption (B) is automatically satisfied if $k+1 < l$. If $k+1 = l$, then Assumption (B) is equivalent to the local concavity at $l$:

$$\sigma_l^2 \ge \frac{1}{2}\left(\sigma_{l+1}^2 + \sigma_{l-1}^2\right).$$

Indeed, as mentioned in Example 2, the condition $\mathcal{P}(\overline{I}) \ge 0$ is more restrictive than local concavity at $l$.

**Theorem 6.2.3.** *Consider a system of competing Brownian particles from Definition* 12. *Fix a subset $J \subseteq \{1, \dots, N-1\}$. Suppose every subset $I$ such that $J \subseteq I \subseteq \{1, \dots, N-1\}$ satisfies assumption (B). Then there a.s. does not exist $t > 0$ such that the system has a multicollision with pattern $J$ at time $t$.*

The following immediate corollary gives a sufficient condition for absence of multicollisions of a given order (and, in particular, multiple collisions of a given order).

**Corollary 6.2.4.** *Consider a classical system of competing Brownian particles from Definition* 12*. Fix an integer* $M = 3, \ldots, N$*, and suppose that every subset* $I \subseteq \{1, \ldots, N-1\}$ *with* $|I| \ge M$ *satisfies condition* (6.11)*. Then a.s. there does not exist* $t > 0$ *such that the system has a multicollision (and, in particular, a collision) of order* $M$

### *6.2.4 Examples of avoiding multicollisions*

In this subsection, we apply Theorem 6.2.3 to systems with a small number of particles: $N = 4$ and $N = 5$. We consider different patterns of multicollisions.

*Example* 6. *Let* $N = 4$ *(four particles) and* $J = \{1, 3\}$*.* (This was already mentioned in the Introduction as Proposition 1.2.3.) A multicollision with pattern $J$ is the same as a simultaneous collision of the following type:

$$Y_1(t) = Y_2(t) \text{ and } Y_3(t) = Y_4(t). \tag{6.13}$$

We need to check Assumption (B) for subsets $I = J = \{1, 3\}$ and $I = \{1, 2, 3\}$. The subset $I = \{1, 2, 3\}$ is a discrete interval. According to Remark 19, we can apply Example 3, and rewrite Assumption (B) as the system of five inequalities (6.7). For $I = \{1, 3\}$, the decomposition (6.10) of $I$ into the union of disjoint non-adjacent discrete intervals has the following form: $I = \{1\} \cup \{3\}$. Therefore, Assumption (B) is always satisfied. Therefore, the system of five inequalities (6.7) is sufficient not only for avoiding total collisions in a system of four particles, but also for avoiding multicollisions (6.13), with pattern $J = \{1, 3\}$.

*Example* 7. *Let* $N = 4$ *and* $J = \{1, 2\}$*.* Let us find a sufficient condition for a.s. avoiding triple collisions of the type $Y_1(t) = Y_2(t) = Y_3(t)$. (This was already mentioned in the Introduction, as Proposition 1.2.4.) There are two subsets $I$ such that $J \subseteq I \subseteq \{1, 2, 3\}$: $I = \{1, 2\}$ and $I = \{1, 2, 3\}$. These two sets are both discrete intervals. As mentioned in the Remark 19, Assumption (B) for $I = \{1, 2, 3\}$ is equivalent to $\mathcal{P}(\overline{I}) \ge 0$, which, in turn, is equivalent to (6.7). Assumption (B) for $I = \{1, 2\}$ is equivalent to local concavity at index 2: $2\sigma_2^2 \ge \sigma_1^2 + \sigma_3^2$. We can write this as the system of six inequalities: local concavity at 2 and the five inequalities (6.7) from Example 3.

*Example* 8. *Consider $N = 5$ (five particles) and take the pattern $J = \{1, 2, 3\}$.* This corresponds to a collision of the following type:

$$Y_1(t) = Y_2(t) = Y_3(t) = Y_4(t). \tag{6.14}$$

There are two subsets $I$ such that $J \subseteq I \subseteq \{1, 2, 3, 4\}$: $I = J = \{1, 2, 3\}$ and $I = \{1, 2, 3, 4\}$. These two sets are both discrete intervals. As mentioned in the Remark 19, Assumption (B) for each of these sets $I$ takes the form $\mathcal{P}(\overline{I}) \ge 0$: $\mathcal{P}(\{1, 2, 3, 4\}) \ge 0$ and $\mathcal{P}(\{1, 2, 3, 4, 5\}) \ge 0$. We can write them as the system of twelve inequalities: the five inequalities (6.7) from Example 3, and the seven inequalities (6.8) from Example 4.

*Example* 9. *Consider $N = 5$ and take the pattern $J = \{1, 2, 4\}$.* This corresponds to a collision

$$Y_1(t) = Y_2(t) = Y_3(t), \quad \text{and} \quad Y_4(t) = Y_5(t). \tag{6.15}$$

There are two subsets $I$ such that $J \subseteq I \subseteq \{1, 2, 3, 4\}$: $I = J = \{1, 2, 3\}$ and $I = \{1, 2, 3, 4\}$. The set $I = \{1, 2, 3, 4\}$ is a discrete interval; by Remark 19, Assumption (B) for $I = \{1, 2, 3, 4\}$ takes the form $\mathcal{P}(\{1, 2, 3, 4, 5\}) \ge 0$. This is equivalent to the conjunction of the seven inequalities (6.8) from Example 4. For $I = \{1, 2, 4\}$, the situation is more complicated. The decomposition of this $I$ into a union of disjoint non-adjacent discrete intervals is $I = \{1, 2\} \cup \{4\}$. So Assumption (B) holds for this set $I$ in one of the following cases:

- if there is local concavity at 2: $\sigma_2^2 \ge \left(\sigma_1^2 + \sigma_3^2\right)/2$;

- Assumption (A) holds for $\{1, 2\}$, which is equivalent to $\mathcal{P}(\{1, 2, 3\}) \ge 0$, which, in turn, is a stronger assumption than local concavity at 2 (see Example 2);

- Assumption (A) holds for $\{4\}$, which is when $\mathcal{P}(\{4, 5\}) \ge 0$; but this is never true, see (6.2);

- Assumption (A) holds for $\{1, 2\} \cup \{4\}$, which is equivalent to

$$\mathcal{T}(\{1, 2, 3\}) + \mathcal{P}(\{4, 5\}) \ge 0, \quad \mathcal{T}(\{4, 5\}) + \mathcal{P}(\{1, 2, 3\}) \ge 0. \tag{6.16}$$

But $\mathcal{P}(\{4,5\}) = \mathcal{P}(\sigma_4, \sigma_5) = -\sigma_4^2 - \sigma_5^2$, as in (6.2), and $\mathcal{P}(\{1,2,3\}) = \mathcal{P}(\sigma_1, \sigma_2, \sigma_3)$ is given by (6.3). Therefore, we have:

$$\mathcal{T}(\{1,2,3\}) + \mathcal{P}(\{4,5\}) = \frac{4}{3}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \sigma_4^2 - \sigma_5^2 \geq 0, \tag{6.17}$$

which can be written as

$$4\sigma_1^2 + 4\sigma_2^2 + 4\sigma_3^2 \geq 3\sigma_4^2 + 3\sigma_5^2. \tag{6.18}$$

The other condition $\mathcal{T}(\{4,5\}) + \mathcal{P}(\{1,2,3\}) \geq 0$ is equivalent to the system of the following three inequalities:

$$\begin{cases} 4\sigma_1^2 + 4\sigma_3^2 \leq 8\sigma_2^2 + 3\sigma_4^2 + 3\sigma_5^2; \\ 4\sigma_1^2 \leq 2\sigma_2^2 + 2\sigma_3^2 + 3\sigma_4^2 + 3\sigma_5^2; \\ 4\sigma_3^2 \leq 2\sigma_1^2 + 2\sigma_2^2 + 3\sigma_4^2 + 3\sigma_5^2. \end{cases} \tag{6.19}$$

Therefore, (6.16) is equivalent to the system of (6.18) and (6.19):

$$\begin{cases} 4\sigma_1^2 + 4\sigma_3^2 \leq 8\sigma_2^2 + 3\sigma_4^2 + 3\sigma_5^2; \\ 4\sigma_1^2 \leq 2\sigma_2^2 + 2\sigma_3^2 + 3\sigma_4^2 + 3\sigma_5^2; \\ 4\sigma_3^2 \leq 2\sigma_1^2 + 2\sigma_2^2 + 3\sigma_4^2 + 3\sigma_5^2; \\ 4\sigma_1^2 + 4\sigma_2^2 + 4\sigma_3^2 \geq 3\sigma_4^2 + 3\sigma_5^2. \end{cases} \tag{6.20}$$

Assumption (B) holds for $I = \{1,2\} \cup \{4\}$ if and only if there is local concavity at 2 or (6.20) hold. Thus, the system of seven inequalities (6.8) from Example 4, together with local concavity at 2 or the four inequalities (6.20), is a sufficient condition for avoiding multicollisions of pattern $\{1,2,4\}$.

*Remark* 20. We can also make use of the condition (6.4) instead of the five inequalities (6.7). If the condition (6.4) is satisfied, then there are a.s. no simultaneous collisions (6.13) at any time $t > 0$. Similarly, in all of the examples involving $N = 4$ particles avoiding certain types of collisions, we can substitute the condition (6.4) instead of the five inequalities (6.7), and the statement will still be true. In Example 7, the two conditions: (6.4) and the local concavity at the index 2, guarantee absence of triple collisions $Y_1(t) = Y_2(t) = Y_3(t)$. The same works for Examples 8 and 9.

*Example* 10. *Suppose we have three or more particles:* $N \ge 3$. Consider the case when all diffusion coefficients are equal to one: $\sigma_1 = \ldots = \sigma_N = 1$. Then there are no triple and multiple collisions, as well as no multicollisions of order $M \ge 3$. To show this, we do not even need to use Theorem 6.2.3. Indeed, using Girsanov transformation as in subsection 5.4.3 of this thesis, see also [103, Subsection 3.2], we can transform the classical system of competing Brownian particles into $N$ independent Brownian motions with zero drifts and unit diffusions. Since the Bessel process of dimension two a.s. does not return to the origin, there are a.s. no triple collisions and multicollisions of order $M \ge 3$ for the system of independent Brownian motions.

Still, we can apply our results to the case of unit diffusion coefficients. Consider total collisions and apply Theorem 6.2.1. Let $\sigma_1 = \ldots = \sigma_N = 1$, so that $\sigma = \mathbf{1} = (1, 1, \ldots, 1)'$; then it is straightforward to calculate that

$$c_l(\sigma) = c_l(\sigma^{\leftarrow}) = 2N - 6,\ l = 1, \ldots, N-1.$$

Therefore, we have:

$$\mathcal{P}(\sigma) = \min(c_1(\sigma),\, \ldots,\, c_{N-2}(\sigma),\, c_{N-1}(\sigma),\, c_1(\sigma^{\leftarrow}),\, \ldots,\, c_{N-2}(\sigma^{\leftarrow})) = 2N - 6 \ge 0.$$

Apply Theorem 6.2.1: the system avoids total collisions. How does this result change if we move the diffusion coefficients $\sigma_1^2, \ldots, \sigma_N^2$ a little away from 1? In other words, if the vector $\sigma$ is in a small neighborhood of $\mathbf{1} = (1, \ldots, 1)' \in \mathbb{R}^N$, what can we say about absence of total collisions?

If $N = 3$, then $\mathcal{P}(\mathbf{1}) = 0$. Even in a small neighborhood of $\mathbf{1}$, we can have either $\mathcal{P}(\sigma) \ge 0$ or $\mathcal{P}(\sigma) < 0$. So we cannot claim that in a certain neighborhood of $\mathbf{1}$ we do not have any total (in this case, triple) collisions. This is consistent with the results of Chapter 5. Indeed, the inequality (5.1) takes the form

$$\sigma_2^2 \ge \frac{1}{2}\left(\sigma_1^2 + \sigma_3^2\right). \tag{6.21}$$

This becomes an equality for $\sigma = (\sigma_1, \sigma_2, \sigma_3)' = \mathbf{1}$. The point $\mathbf{1}$ lies at the boundary of the set of points in $\mathbb{R}^3$ given by (6.21). Or, equivalently, in any neighborhood of $\mathbf{1}$ there are both points $\sigma$ which satisfy (6.21) and which do not satisfy (6.21).

But for $N \ge 4$ (four or more particles), we have: $\mathcal{P}(\mathbf{1}) > 0$. Since $\mathcal{P}(\sigma)$ is a continuous function of $\sigma$, there exists a neighborhood $\mathcal{U}$ of $\mathbf{1}$ such that for all $\sigma \in \mathcal{U}$ we have: $\mathcal{P}(\sigma) > 0$, and the system of competing Brownian particles does not have total collisions.

## 6.3 Results and Proofs for an SRBM in the Orthant

### 6.3.1 Statements of results

There are three important theorems. First, we provide a sufficient condition for not hitting the corner, and another sufficient condition for hitting the corner. Taken together, they do not give us a necessary and sufficient condition, because there is a gap between them. In this respect, these results are different from that of Chapter 5, where we gave a necessary and sufficient condition for avoiding non-smooth parts of the boundary.

A remaining question is about hitting or avoiding a given edge $S_I$ of the boundary $\partial S$. We provide another theorem which reduces it to the question of not hitting the corner. This gives us a sufficient condition for not hitting the given edge of $\partial S$.

The last of these three main results is a sufficient condition for hitting a given edge of $\partial S$.

**Definition 31.** We say that the matrix $R$ which is a reflection nonsingular $\mathcal{M}$-matrix *satisfies Assumption (C)* if there exists a diagonal $d \times d$-matrix $C = \mathrm{diag}(c_1, \ldots, c_d)$ with $c_i > 0$ such that $\overline{R} = RC$ is a symmetric matrix.

We denote $R^{-1} = (\rho_{ij})_{1 \le i,j \le d}$, and consider the following constants:

$$c_+ := \max_{x \in S \setminus \{0\}} \frac{x' R^{-1} A R^{-1} x}{x' R^{-1} x}, \qquad c_- := \min_{x \in S \setminus \{0\}} \frac{x' R^{-1} A R^{-1} x}{x' R^{-1} x}.$$

**Lemma 6.3.1.** *These numbers $c_\pm$ are well defined and strictly positive.*

The (rather straightforward) proof is postponed until the Appendix. The following theorem is our main result about an SRBM hitting the corner.

**Theorem 6.3.2.** *Suppose the matrix $R$ satisfies Assumption (C).*

*(i) If the following condition is true:*

$$\operatorname{tr}\left(R^{-1}A\right) \ge 2c_+, \tag{6.22}$$

*then the* $\mathrm{SRBM}^d(R, \mu, A)$ *does not hit the corner.*

*(ii) If the following condition is true:*

$$0 \le \operatorname{tr}\left(R^{-1}A\right) < 2c_-, \tag{6.23}$$

*then the* $\mathrm{SRBM}^d(R, \mu, A)$ *hits the corner.*

Sometimes the numbers $c_\pm$ are difficult to calculate. Let us give useful estimates of $c_+$ from above, and of $c_-$ from below.

**Lemma 6.3.3.** *Suppose the matrix $R$ satisfies Assumption (C). If, in addition, $\rho_{ij} > 0$ for all $i, j = 1, \ldots, d$, then*

$$c_+ \le \overline{c}_+ := \max_{1 \le i \le j \le d} \frac{(R^{-1}AR^{-1})_{ij}}{\rho_{ij}}, \quad c_- \ge \overline{c}_- := \min_{1 \le i \le j \le d} \frac{(R^{-1}AR^{-1})_{ij}}{\rho_{ij}}.$$

The next theorem establishes a connection between not hitting the corner and not hitting an edge. It is similar to results from [59], and we took the proof technique from [59].

**Theorem 6.3.4.** *Consider an* $\mathrm{SRBM}^d(R, \mu, A)$*. Fix a nonempty subset $J \subseteq \{1, \ldots, d\}$. Suppose for every $I$ such that $J \subseteq I \subseteq \{1, \ldots, d\}$ the process* $\mathrm{SRBM}^{|I|}([R]_I, [\mu]_I, [A]_I)$ *does not hit the corner. Then an* $\mathrm{SRBM}^d(R, \mu, A)$ *does not hit the edge $S_I$.*

The last of our main results about SRBM links hitting corners to hitting edges.

**Theorem 6.3.5.** *Consider an* $\mathrm{SRBM}^d(R, \mu, A)$ *with a reflection nonsingular* $\mathcal{M}$*-matrix* $R$. *Fix a nonempty subset* $I \subseteq \{1, \dots, d\}$. *Suppose an* $\mathrm{SRBM}^{|I|}([R]_I, [\mu]_I, [A]_I)$ *hits the corner. Then an* $\mathrm{SRBM}^d(R, \mu, A)$ *hits the edge* $S_I$.

This theorem is proved using *stochastic comparison*: it trivially follows from Proposition 4.3.4.

The rest of the section will be devoted to the proofs of Theorems 6.3.2 and 6.3.4.

### *6.3.2 Proof of Theorem* 6.3.2

First, we present an informal overview of the proof, and then give a complete proof.

#### *Outline of the proof*

Let $Z = (Z(t), t \ge 0)$ be an $\mathrm{SRBM}^d(R, \mu, A)$, starting from $z \in S$. By Proposition 5.3.1, we can assume $z \in S \setminus \partial S$, and $\mu = 0$. Consider the function

$$F(x) := x' R^{-1} x. \tag{6.24}$$

Since the matrix $R$ is a reflection nonsingular $\mathcal{M}$-matrix, by Lemma 2.2.1 from Chapter 2, which corresponds to [103, Lemma 2.1] the matrix $R^{-1}$ has all elements nonnegative: $\rho_{ij} \ge 0$, with strictly positive elements on the main diagonal: $\rho_{ii} > 0$, $i = 1, \dots, d$. Therefore, if $F(x) = 0$ for a certain $x \in S$, then $x = 0$. The process $Z$ hits the corner if and only if the process $F(Z(\cdot))$ hits zero. Let $L = (L(t), t \ge 0)$ be the vector of regulating processes for $Z$, and let $B = (B(t), t \ge 0)$ be the driving Brownian motion for $Z$, so that we have:

$$Z(t) = B(t) + RL(t),\ t \ge 0. \tag{6.25}$$

We see that the process $Z$ has a diffusion term and a regulating process term. The reason for applying the function $F$ to this process is that, if we write an equation for $F(Z(\cdot))$ using the Itô-Tanaka formula, the terms corresponding to the regulating processes vanish, and $F(Z(\cdot))$ is an Itô process.

It turns out that its drift coefficient is constant and its diffusion coefficient is comparable with that in the SDE for Bessel squared process. After an appropriate random time-change, we can make the diffusion coefficient exactly equal to the one for a Bessel squared process. However, this will not turn our process into a Bessel squared process. Indeed, the drift coefficient for the new process will not be constant (and for a Bessel squared process, it is constant). Still, we can bound this drift coefficient by a constant, which allows to compare the new time-changed process with a Bessel squared process. But we know that a Bessel squared process hits zero if and only if its index is less than two.

This allows us to find whether the process $F(Z(\cdot))$ hits or does not hit zero. This, in turn, is equivalent to whether the process $Z$ hits the origin.

*Complete proof*

By Lemma 2.2.1, which corresponds to [103, Lemma 2.1] (equivalent characterization of reflection nonsingular $\mathcal{M}$-matrices), we have:

$$\left(R^{-1}\right)_{ij} \geq 0, \quad i,j = 1,\dots,d; \quad \left(R^{-1}\right)_{ii} > 0, \quad i = 1,\dots,d.$$

Therefore, the matrix $R^{-1} = C^{-1}R^{-1} = (\rho_{ij})_{1\leq i,j\leq d}$ has elements $\rho_{ij} = c_i^{-1}\left(R^{-1}\right)_{ij}$. By Assumption 31, the matrix $R^{-1}$ is symmetric. Therefore, its entries satisfy

$$\rho_{ij} = \rho_{ji} \geq 0, \quad i,j = 1,\dots,d; \quad \rho_{ii} > 0, \quad i = 1,\dots,d. \tag{6.26}$$

Recall the definition of function $F$ from (6.24). From (6.26) we have: $F(x) > 0$ for $x \in S\setminus\{0\}$. Since the matrix $R^{-1}$ is symmetric, the first and second order derivatives of the function $F$ are

$$\frac{\partial F}{\partial x_i} = \left(2R^{-1}x\right)_i = 2\sum_{k=1}^{d}\rho_{ik}x_k, \quad \frac{\partial^2 F}{\partial x_i \partial x_j} = 2\rho_{ij}, \quad i,j = 1,\dots,d.$$

Note that $\langle Z_i, Z_j\rangle_t = \langle B_i, B_j\rangle_t = a_{ij}t$. By the Itô-Tanaka formula applied to the process $Z$ from (6.25) and the function $F$ from (6.24), we have:

$$\mathrm{d}F(Z(t)) = \sum_{i=1}^{d}\frac{\partial F}{\partial x_i}(Z(t))\mathrm{d}Z_i(t) + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial^2 F}{\partial x_i \partial x_j}(Z(t))\mathrm{d}\langle Z_i, Z_j\rangle_t \tag{6.27}$$

$$= \sum_{i=1}^{d} \big(2R^{-1}Z(t)\big)_i \mathrm{d}B_i(t) + \sum_{i=1}^{d}\sum_{k=1}^{d} \big(2R^{-1}Z(t)\big)_i r_{ik}\mathrm{d}L_k(t) + \sum_{i=1}^{d}\sum_{j=1}^{d} \rho_{ij}a_{ij}\mathrm{d}t \tag{6.28}$$

$$= 2\sum_{i=1}^{d}\sum_{j=1}^{d} \rho_{ij}Z_j(t)\mathrm{d}B_i(t) + 2\sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{k=1}^{d} \rho_{ij}Z_j(t)r_{ik}\mathrm{d}L_k(t) + \mathrm{tr}\big(R^{-1}A\big)\mathrm{d}t. \tag{6.29}$$

For each $j = 1, \ldots, d$, the regulating process $L_j$ can grow only if $Z_j = 0$: we express this by writing $Z_j(t)\mathrm{d}L_j(t) = 0$. Using this, we shall now show that the second term in (6.29) is actually equal to zero:

$$\begin{aligned}
\sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{k=1}^{d}\rho_{ij}Z_j(t)r_{ik}\mathrm{d}L_k(t) &= \sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{k=1}^{d} \rho_{ji}Z_j(t)r_{ik}\mathrm{d}L_k(t) \\
&= \sum_{j=1}^{d}\sum_{k=1}^{d}\big(R^{-1}R\big)_{jk}Z_j(t)\mathrm{d}L_k(t) = \sum_{j=1}^{d}\sum_{k=1}^{d}\big(C^{-1}I_d\big)_{jk}Z_j(t)\mathrm{d}L_k(t) \\
&= \sum_{j=1}^{d}\sum_{k=1}^{d} c_j^{-1}\delta_{jk}Z_j(t)\mathrm{d}L_k(t) = \sum_{j=1}^{d} c_j^{-1}Z_j(t)\mathrm{d}L_j(t) = 0.
\end{aligned}$$

Therefore, the process $F(Z(\cdot))$ does not have terms corresponding to the regulating processes. Instead, $F(Z(\cdot))$ has only an absolutely continuous term and a local martingale term: this is an Itô process.

$$\mathrm{d}F(Z(t)) = 2\sum_{i=1}^{d}\sum_{j=1}^{d} \rho_{ij}Z_j(t)\mathrm{d}B_i(t) + \mathrm{tr}\big(R^{-1}A\big)\mathrm{d}t. \tag{6.30}$$

Recall that $B_1, \ldots, B_d$ are driftless one-dimensional Brownian motions (they are driftless, because the drift $\mu = 0$, according to our assumptions). Therefore, the following process is a continuous local martingale:

$$M = (M(t), t \ge 0), \quad M(t) := 2\sum_{i=1}^{d}\sum_{j=1}^{d}\int_0^t \rho_{ij}Z_j(s)\mathrm{d}B_i(s).$$

So we can rewrite (6.30) as

$$F(Z(t)) = F(z) + M(t) + \mathrm{tr}\big(R^{-1}A\big)t.$$

Let us calculate the quadratic variation of $M$. It turns out to be comparable with that of a Bessel squared process. Then we make a time-change to transform $F(Z(\cdot))$ into a process which can be compared to a Bessel squared process. Recall that, by definition of the process $B$, $\langle B_i, B_j\rangle_t = a_{ij}t$. Let

$$M_{ij}(t) = \int_0^t \int_0^t Z_j(s)\rho_{ij}\mathrm{d}B_i(s), \quad i, j = 1, \ldots, d.$$

For $i, j, k, l = 1, \ldots, d$, we have:

$$\langle M_{ij}, M_{kl}\rangle_t = \int_0^t Z_j(s)\rho_{ij}Z_l(s)\rho_{kl}a_{ik}\mathrm{d}s.$$

But the quadratic variation of $M = \sum_{i=1}^d\sum_{j=1}^d M_{ij}$ is equal to the sum

$$\begin{aligned}\langle M\rangle_t &= \sum_{i=1}^d\sum_{j=1}^d\sum_{k=1}^d\sum_{l=1}^d\langle M_{ij}, M_{kl}\rangle_t = \sum_{i=1}^d\sum_{j=1}^d\sum_{k=1}^d\sum_{l=1}^d\int_0^t Z_j(s)\rho_{ij}Z_l(s)\rho_{kl}a_{ik}\mathrm{d}s \\ &= \sum_{i=1}^d\sum_{j=1}^d\sum_{k=1}^d\sum_{l=1}^d\int_0^t Z_j(s)\rho_{ij}a_{ik}\rho_{kl}Z_l(s)\mathrm{d}s = \int_0^t \left(Z'(s)R^{-1}AR^{-1}Z(s)\right)\mathrm{d}s.\end{aligned}$$

Let $\tau = \inf\{t \ge 0 \mid Z(t) = 0\}$ be the first moment when the process $Z$ hits the corner. Since $z = Z(0) \in S \setminus \partial S$, we have: $\tau > 0$ a.s. Let

$$q(s) := \left(Z'(s)R^{-1}AR^{-1}Z(s)\right)^{1/2}, \quad s \ge 0.$$

Then we can represent $M$ as the stochastic integral

$$M(t) = 2\int_0^t q(s)\mathrm{d}\overline{W}(s),$$

where $\overline{W} = (\overline{W}(t), t \ge 0)$ is a standard Brownian motion; and for $s < \tau$, we have $Z(s) \in S \setminus \{0\}$, and $F(Z(s)) > 0$. It follows from the definition of constants $c_\pm$ that

$$\frac{1}{2}c_-^{1/2} \le \frac{q(s)}{2F^{1/2}(Z(s))} = \frac{1}{2}\left(\frac{Z'(s)R^{-1}AR^{-1}Z(s)}{Z'(s)R^{-1}Z(s)}\right)^{1/2} \le \frac{1}{2}c_+^{1/2}.$$

Make the following time change:

$$\Delta(t) := \int_0^t \frac{q^2(s)}{4F(Z(s))}\mathrm{d}s, \quad t \le \tau.$$

By [99, Lemma 2], this is a strictly increasing function on $[0, \tau]$ with $\Delta(0) = 0$. Denote $s_0 := \Delta(\tau)$. Define the inverse of $\Delta$ by

$$\chi(s) := \inf\{t \geq 0 \mid \Delta(t) \geq s\}.$$

The following process will be compared with Bessel squared process:

$$V(s) \equiv F(Z(\chi(s))), \quad s \in [0, s_0].$$

By [99, Lemma 2], the process $V = (V(s), s \geq 0)$ satisfies the following equation:

$$\begin{aligned} \mathrm{d}V(s) &= \mathrm{tr}\big(R^{-1}A\big)\frac{V(s)}{q^2(\chi(s))}\mathrm{d}s + v(F(Z(s)))\mathrm{d}W(s) \\ &= \mathrm{tr}\big(R^{-1}A\big)\frac{V(s)}{q^2(\chi(s))}\mathrm{d}s + 2V^{1/2}(s)\mathrm{d}W(s). \end{aligned}$$

Here, $W = (W(t), t \geq 0)$ is yet another standard Brownian motion. Note that

$$\frac{1}{4}c_- \leq \Delta'(s) = \frac{q^2(s)}{4F(Z(s))} = \frac{Z'(s)\overline{R}^{-1}A\overline{R}^{-1}Z(s)}{4Z'(s)\overline{R}^{-1}Z(s)} \leq \frac{1}{4}c_+.$$

So the mapping $\Delta : [0, \tau) \to [0, s_0)$ is one-to-one, and $\tau = \infty$ if and only if $s_0 = \infty$. Then we have:

$$\mathbf{P}\left(\exists t > 0 : F(Z(t)) = 0\right) = 0 \ \text{ if and only if } \ \mathbf{P}\left(\exists s > 0 : V(s) = 0\right) = 0.$$

Suppose the condition (6.22) holds. We need to prove that the process $Z$ does not hit the corner. Assume the converse. Then $\mathbf{P}(\tau < \infty) > 0$, so $\mathbf{P}(s_0 < \infty) > 0$. On the event $\{s_0 < \infty\}$, we have: $V(s_0) = 0$. Note that

$$\frac{V(s)}{q^2(\chi(s))} \geq c_+^{-1},$$

and $\mathrm{tr}(R^{-1}A) \geq 2c_+ \geq 0$, so

$$\mathrm{tr}\big(R^{-1}A\big)\frac{V(s)}{q^2(\chi(s))} \geq \mathrm{tr}\big(R^{-1}A\big)c_+^{-1} =: \beta \geq 2.$$

Consider the squared Bessel process $\overline{V} = (\overline{V}(s), s \geq 0)$, given by the equation

$$\mathrm{d}\overline{V}(s) = 2\overline{V}^{1/2}(s)\mathrm{d}W(s) + \beta\mathrm{d}s, \quad \overline{V}(0) = V(0).$$

Since $\beta \geq 2$, it is known (see, e.g., [97, Section 11.1, p. 442]) that $\overline{V}$ a.s. does not hit 0. By standard comparison theorems, see for example [61, Chapter 6, Theorem 1.1], we have: $V(s) \geq \overline{V}(s)$ a.s. for $s < s_0$. So if $s_0 < \infty$, then by continuity $V(s_0) \geq \overline{V}(s_0) > 0$, but $V(s_0) = 0$. This contradiction completes the proof of (i). The proof of (ii) is similar.

### *6.3.3 Proof of Theorem* 6.3.4

We prove this theorem using induction by $d - |I|$.
Induction base: $d - |I| = 0$, then $I = \{1, \dots, d\}$, and the statement is trivial.
Induction step: fix $q = 0, 1, 2, \dots$ and suppose the statement is true for $d - |I| = q - 1$; then prove it for $d - |I| = q$.

For $\varepsilon \in (0, 1)$, let $K_\varepsilon = \{x \in S \mid \varepsilon \leq \|x\| \leq \varepsilon^{-1}\}$. Fix a point $z \in S \setminus \{0\}$, so that $z \in K_\varepsilon$ for all $\varepsilon > 0$ small enough. Start a copy of an $\mathrm{SRBM}^d(R, \mu, A)$ from $z$. Denote this copy by $Z = (Z(t), t \geq 0)$, and let $B = (B(t), t \geq 0)$ be its driving Brownian motion. Let

$$\tau := \inf\{t \geq 0 \mid Z(t) \in S_I\}$$

be the first moment when the process $Z$ hits the edge $S_I$. We need to show that $\tau = \infty$ a.s. Let

$$\eta_\varepsilon := \inf\{t \geq 0 \mid Z(t) \in K_\varepsilon\}.$$

Note that $\eta_\varepsilon \leq \eta_{\varepsilon'}$ when $\varepsilon' \leq \varepsilon$, and $\lim_{\varepsilon\downarrow 0} \eta_\varepsilon = \infty$, because by assumptions of the theorem the process $Z$ does not hit the corner: $Z(t) \neq 0$ for all $t \geq 0$ a.s. So it suffices to show that $\tau \geq \eta_\varepsilon$ for all $\varepsilon \in (0, 1)$. Fix an $\varepsilon \in (0, 1)$. For every $x \in K_\varepsilon$, there exists an open neighborhood $U(x)$ of $x$ with the following property: there exists some index $i = i(x) \in \{1, \dots, d\}$ such that for all $y \in U(x)$ we have: $y_{i(x)} > 0$. Since $K_\varepsilon$ is compact, we can extract a finite subcover $U(x_1), \dots, U(x_s)$. Without loss of generality, let us include the neighborhood $U(x_0)$ of $x_0 = z$ into this subcover. Now, define a sequence of stopping times:

$$\tau_0 := 0, \quad j_0 := 0; \quad \tau_{k+1} := \inf\{t \geq \tau_k \mid Z(t) \notin U\left(x_{j_k}\right)\},$$

and $j_{k+1}$ is defined as any $j = 0, \dots, s$ such that $Z(\tau_{k+1}) \in U(x_j)$. Suppose that, at some point, we cannot find such $j$; in other words,

$$Z(\tau_{k+1}) \notin U(x_{j_0}) \cup U(x_{j_1}) \cup \ldots \cup U(x_{j_s}).$$

Then the sequence of stopping times terminates, and we denote $K := k+1$. In this case, we have defined $\tau_0, j_0, \tau_1, j_1, \dots, \tau_{K-1}, j_{K-1}, \tau_K$. If the sequence does not terminate, we let $K = \infty$. So we have:

$$Z_{j_k}(t) > 0 \ \text{ for } \ t \in [\tau_k, \tau_{k+1}), \quad k < K.$$

The sequence $(\tau_k)$ can be either finite or countable. Recall that $U(x_j),\ j = 0, \dots, s$ is a cover of $K_\varepsilon$. Therefore, $\sup_k \tau_k \ge \eta_\varepsilon$. It suffices to show that $\tau \ge \tau_k$. We prove this using induction by $k$.

Base: $k = 1$. If $j_0 \in I$, then $Z_{j_0}(t) > 0$ for $t < \tau_1$, and so $Z(t) \notin S_I$. In this case, $\tau \ge \tau_1$ is straightforward. Now, if $j_0 \notin I$, then consider the set $J := \{1, \dots, d\} \setminus \{j_0\}$. We have the following representation:

$$([Z(t \wedge \tau_1)]_J,\ t \ge 0) = (\overline{Z}(t \wedge \tau_1),\ t \ge 0),$$

where $\overline{Z} = (\overline{Z}(t), t \ge 0)$ is an $\mathrm{SRBM}^{d-1}([R]_J, [\mu]_J, [A]_J)$, starting from $[z]_J$, with the driving Brownian motion $[B]_J = ([B(t)]_J, t \ge 0)$. This process $\overline{Z}$ is well defined, since the matrix $[R]_J$ is a reflection nonsingular $\mathcal{M}$-matrix, and by Proposition 2.3.1 there exists a strong version of $\overline{Z}$. So by the induction hypothesis, a.s. there does not exist $t \ge 0$ such that $\overline{Z}(t) \in S_I$, because $d - 1 - |I| = q - 1$. For every $y \in S$, we have: $y \in S_I$ if and only if $[y]_J \in S_I$. Therefore, for all $t < \tau_1$ we have: $Z(t) \notin S_I$. This proves that $\tau \ge \tau_1$.

Induction step: suppose $t \ge \tau_k$ and $k < K$, that is, the sequence does not terminate at this step. Then we need to prove $\tau \ge \tau_{k+1}$. Consider the process $(Z(t + \tau_k), t \ge 0)$. This is a version of an $\mathrm{SRBM}^d(R, \mu, A)$, started from $Z(\tau_k)$. But

$$Z(\tau_k) \in U(x_{j_0}) \cup U(x_{j_1}) \cup \ldots \cup U(x_{j_s}).$$

There exists $j = 0, \dots, s$ such that $Z(\tau_k) \in U(x_j)$. In addition, $Z(\tau_k) \in S \setminus \{0\}$, because by induction hypothesis, the process $Z$ never hits the corner. Apply the reasoning from the induction base to this process instead of the original SRBM. The moment $\tau_{k+1} - \tau_k$ plays the role of $\tau_1$ above, and the moment $\tau - \tau_k$ plays the role of $\tau$. So $\tau - \tau_k \ge \tau_{k+1} - \tau_k$, and $\tau \ge \tau_{k+1}$. This completes the proof.

### *6.3.4 Corollaries of the main results for an SRBM*

The following corollary of Theorem 6.3.4 gives a sufficient condition for not hitting edges of a given order.

**Corollary 6.3.6.** *Consider an* $\mathrm{SRBM}^d(R, \mu, A)$. *Fix* $p = 2, \dots, d-1$. *Suppose for every* $I \subseteq \{1, \dots, d\}$ *such that* $|I| \ge p$ *the process* $\mathrm{SRBM}^{|I|}([R]_I, [\mu]_I, [A]_I)$ *does not hit the corner. Then an* $\mathrm{SRBM}^d(R, \mu, A)$ *does not hit edges of order* $p$.

The next corollary combines the results of Theorem 6.3.2, Theorem 6.3.4 and Theorem 6.3.5. Its proof is trivial and is omitted.

**Corollary 6.3.7.** *Take an* $\mathrm{SRBM}^d(R, \mu, A)$. *Suppose the matrix* $R$ *satisfies Assumption* 31.

*(i) Fix a nonempty subset* $J \subseteq \{1, \dots, d\}$. *Suppose that for every subset* $I$ *such that* $J \subseteq I \subseteq \{1, \dots, d\}$ *we have:*

$$\mathrm{tr}\left([\overline{R}]_I^{-1}[A]_I\right) \ge 2 \max_{x \in \mathbb{R}_+^{|I|} \setminus \{0\}} \frac{x'[\overline{R}]_I^{-1}[A]_I[\overline{R}]_I^{-1}x}{x'[\overline{R}]_I^{-1}x}. \tag{6.31}$$

*Then the* $\mathrm{SRBM}^d(R, \mu, A)$ *avoids* $S_I$.

*(ii) Fix* $p = 1, \dots, d-1$. *Suppose for every subset* $I \subseteq \{1, \dots, d\}$ *with* $|I| \ge p$ *we have:*

$$\mathrm{tr}\left([\overline{R}]_I^{-1}[A]_I\right) \ge 2 \max_{x \in \mathbb{R}_+^{|I|} \setminus \{0\}} \frac{x'[\overline{R}]_I^{-1}[A]_I[\overline{R}]_I^{-1}x}{x'[\overline{R}]_I^{-1}x}.$$

*Then the* $\mathrm{SRBM}^d(R, \mu, A)$ *avoids edges of order* $p$.

*(iii) Suppose there exists a subset* $I \subseteq \{1, \dots, d\}$ *such that*

$$\mathrm{tr}\left([\overline{R}]_I^{-1}[A]_I\right) < 2 \min_{x \in \mathbb{R}_+^{|I|} \setminus \{0\}} \frac{x'[\overline{R}]_I^{-1}[A]_I[\overline{R}]_I^{-1}x}{x'[\overline{R}]_I^{-1}x}.$$

*Then the* $\mathrm{SRBM}^d(R, \mu, A)$ *hits* $S_I$.

## 6.4 Proofs of Theorems 6.2.1 and 6.2.3

### 6.4.1 Outline of the proofs

Consider a system of competing Brownian particles from Definition 12. In Lemma 6.4.1, we note that a multicollision with pattern $I$ is equivalent to an $\mathrm{SRBM}^{N-1}(R, \mu, A)$ hitting the edge $S_I$ of the $N-1$-dimensional orthant $\mathbb{R}_+^{N-1}$. Here, the parameters $R$, $\mu$, $A$ are given by (3.16), (3.9) and (3.8) below. We apply Theorem 6.3.2 and Theorem 6.3.4 to this SRBM to prove Theorems 6.2.1 and 6.2.3 respectively. We use the estimate in Lemma 6.3.3 for $c_+$, since the right-hand side of (6.22) seems hard to compute for matrices $R$ and $A$ given by (3.16) and (3.8).

Since the matrix $R$ from (3.16) is itself symmetric, we can take $C = I_{N-1}$ and $\overline{R} = R$. The inverse matrix $R^{-1} = \overline{R}^{-1} = (\rho_{ij})_{1\le i,j\le N-1}$ has the form

$$\rho_{ij} = \begin{cases} 2i(N-j)/N, \ i \le j; \\ 2j(N-i)/N, \ i \ge j \end{cases} \tag{6.32}$$

This result can be found in [13] or [56] (the latter article deals with a slightly different matrix, from which one can easily find the inverse of the given matrix $R$).

After a (rather tedious) computation, we rewrite the condition (6.22) from Theorem 6.3.2 as $\mathcal{P}(\sigma) \ge 0$, where $\mathcal{P}(\sigma)$ is defined in (6.1). This proves Theorem 6.2.1.

Proving Theorem 6.2.3 is a bit harder. Apply Theorem 6.3.4, and fix a subset $I \subseteq \{1, \ldots, N-1\}$ such that $J \subseteq I$. We need to find a sufficient condition for an

$$\mathrm{SRBM}^{|I|}([R]_I, [\mu]_I, [A]_I)$$

to a.s. avoid the corner of the orthant $\mathbb{R}_+^{|I|}$. We decompose the set $I$ as in (6.10):

$$I = I_1 \cup I_2 \cup \ldots I_r,$$

into a union of disjoint non-adjacent discrete intervals. In Lemma 6.4.7, we prove that if $I$ satisfies Assumption (B), then the $\mathrm{SRBM}^{|I|}([R]_I, [\mu]_I, [A]_I)$ indeed avoids the corner. This completes the proof of Theorem 6.2.3. But to prove Lemma 6.4.7, we need to consider different variants of decomposition (6.10). For example, if $I_1 = \{1\}$ and $I_2 = \{3\}$, then this guarantees that an $\mathrm{SRBM}^{|I|}([R]_I, [\mu]_I, [A]_I)$ avoids the corner. Various cases are considered in Lemmas 6.4.8, 6.4.9 and 6.4.10, which constitute the crux of the proof.

### *6.4.2 Collisions of particles and the gap process*

The following lemma translates statements about multiple collisions and multicollisions of competing Brownian particles to the language of an SRBM. The proof is trivial and is therefore omitted.

**Lemma 6.4.1.** *Consider a classical system of $N$ competing Brownian particles. Then there is a multicollision with pattern $I$ at time $t$ if and only if the gap process hits the edge $S_I$ at time $t$. For example, there is a total collision at time $t$ if and only if the gap process hits the corner at time $t$.*

For example, $Y_1(t) = Y_2(t)$ and $Y_3(t) = Y_4(t) = Y_5(t)$ is a multicollision of order 3, with pattern $\{1, 3, 4\}$, which is equivalent of the gap process hitting the edge $\{z_1 = z_3 = z_4 = 0\}$. Similarly, $Y_3(t) = Y_4(t) = Y_5(t) = Y_6(t)$ is a collision of order 3 (which is also a particular case of a multicollision of order 3, with pattern $\{3, 4, 5\}$), and it is equivalent to the gap process hitting the edge $\{z_3 = z_4 = z_5 = 0\}$.

### *6.4.3 Avoiding a multicollision depends only on diffusion coefficients*

The following lemma tells us that the property of a system of competing Brownian particles to avoid multicollisions with a given pattern is independent of the initial conditions $x$ and the drift coefficients $g_1, \ldots, g_N$. In other words, it can possibly depend only on the diffusion coefficients $\sigma_1^2, \ldots, \sigma_N^2$.

**Lemma 6.4.2.** *Take a classical system of competing Brownian particles from Definition* 12. *Fix $I \subseteq \{1, \ldots, N-1\}$, a pattern. Let $x \in \mathbb{R}^N$ be the initial conditions, and let $\mathbf{P}_x$ be the* *corresponding probability measure. Denote by*

$$p\,(g_1,\,g_2,\,\ldots,\,g_N,\,\sigma_1,\,\sigma_2,\,\ldots\,\sigma_N,\,x) \tag{6.33}$$

*the probability that there exists a moment $t > 0$ such that the system, starting from $x$, will experience a multicollision with pattern $I$ at this moment. For fixed $\sigma_1, \ldots, \sigma_N > 0$, either*

$$p\,(g_1,\,g_2,\,\ldots,\,g_N,\,\sigma_1,\,\sigma_2,\,\ldots\,\sigma_N,\,x) = 0 \ \ \textit{for all} \ \ x \in \mathbb{R}^N, \quad (g_k)_{1\le k\le N} \in \mathbb{R}^N,$$

*or*

$$p\,(g_1,\,g_2,\,\ldots,\,g_N,\,\sigma_1,\,\sigma_2,\,\ldots\,\sigma_N,\,x) > 0 \ \ \textit{for all} \ \ x \in \mathbb{R}^N, \quad (g_k)_{1\le k\le N} \in \mathbb{R}^N.$$

However, in the second case (when the probability (6.33) is positive) the exact value of this probability depends on the initial conditions $x$ and the drift coefficients $g_1, \ldots, g_N$. This follows from Remark 15 from Chapter 5, which corresponds to [103, Subsection 3.2, Remark 5] and connection between competing Brownian particles and an SRBM, discussed just above.

*Proof.* Follows from Lemma 2.2.1 ([103, Lemma 3.1]), and the reduction of multicollisions to hitting edges of the orthant which was done right above. □

### *6.4.4 Some preliminary calculations*

As mentioned before, the matrix $R$ in (3.16) is itself symmetric, so we take $C = I_{N-1}$, and $\overline{R} = R$. Without loss of generality, let

$$\rho_{ij} = 0, \quad i = 0,\ N,\ j = 0, \ldots, N \text{ or } j = 0,\ N,\ i = 0, \ldots, N.$$

This is consistent with the notation (6.32). Note that $\rho_{ij} > 0$ for $i, j = 1, \ldots, N-1$: all elements of the matrix $R^{-1}$ are positive. Therefore, we can apply an estimate from Lemma 6.3.3:

$$c_+ := \max_{x\in\mathbb{R}^{N-1}\setminus\{0\}} \frac{x'R^{-1}AR^{-1}x}{x'R^{-1}x} \le \max_{1\le k\le l\le N-1} \frac{\left(R^{-1}AR^{-1}\right)_{kl}}{\rho_{kl}}.$$

**Lemma 6.4.3.** *For the matrix $R$ given by* (3.16) *and the matrix $A$ given by* (3.8)*, we have in the notation of* (6.9)*:*

$$\operatorname{tr}\bigl(R^{-1}A\bigr) = \mathcal{T}(\sigma)\,. \tag{6.34}$$

*Proof.* Straightforward calculation gives

$$\begin{aligned}
\operatorname{tr}\bigl(R^{-1}A\bigr) &= \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\rho_{ij}a_{ij} = \sum_{i=1}^{N-1}(\sigma_i^2+\sigma_{i+1}^2)\frac{2i(N-i)}{N} \\
&+ 2\sum_{i=2}^{N-1}(-\sigma_i^2)\frac{2(i-1)(N-i)}{N} = \frac{2(N-1)}{N}\sigma_1^2 + \frac{2(N-1)}{N}\sigma_N^2 \\
&+ \sum_{k=2}^{N-1}\sigma_k^2\left(\frac{2k(N-k)}{N} + \frac{2(k-1)(N-k+1)}{N} - 2\frac{2(k-1)(N-k)}{N}\right) \\
&= \frac{2(N-1)}{N}\sum_{k=1}^{N}\sigma_k^2 = \mathcal{T}(\sigma).
\end{aligned}$$

□

The following lemma helps us simplify the matrix $R^{-1}AR^{-1}$, where $A$ is given by (3.8), and $R^{-1}$ is given by (6.32).

**Lemma 6.4.4.** *Consider the matrix $A$ as in* (3.8)*, and take a symmetric $(N-1)\times(N-1)$-matrix $Q=(q_{ij})$. Augment it by two additional rows and two additional columns, one from each side, and fill them with zeros:*

$$q_{ij} = 0 \quad \textit{for} \quad i = 0,\ N,\ \ j = 0,\dots,N,\ \ \textit{and for} \quad j = 0,\ N,\ \ i = 0,\dots,N.$$

*Then for $k,l = 1,\dots,N-1$ we have:*

$$(QAQ)_{kl} = \sum_{p=1}^{N}\left(q_{pk}-q_{p-1,k}\right)\left(q_{pl}-q_{p-1,l}\right)\sigma_p^2.$$

*Proof.* The matrix $A$ is tridiagonal:

$$\begin{cases}
a_{ii} = \sigma_i^2 + \sigma_{i+1}^2,\ \ i = 1,\dots,N-1; \\
a_{i,i+1} = a_{i+1,i} = -\sigma_{i+1}^2,\ \ i = 1,\dots,N-2; \\
a_{ij} = 0,\quad i,j = 1,\dots,N-1,\ \ |i-j| \ge 2.
\end{cases}$$

Using the symmetry of $Q$, we have:

$$\begin{aligned}(QAQ)_{kl} &= \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} q_{ik}q_{jl}a_{ij} = \sum_{p=1}^{N-1}\left(\sigma_p^2+\sigma_{p+1}^2\right)q_{pk}q_{pl} - \sum_{p=2}^{N-1}\sigma_p^2 q_{pk}q_{p-1,l} - \sum_{p=2}^{N-1}\sigma_p^2 q_{p-1,k}q_{pl}\\
&= \sum_{p=1}^{N}\sigma_p^2 q_{pk}q_{pl} + \sum_{p=1}^{N}\sigma_p^2 q_{p-1,k}q_{p-1,l} - \sum_{p=1}^{N}\sigma_p^2 q_{pk}q_{p-1,l} - \sum_{p=1}^{N}\sigma_p^2 q_{p-1,k}q_{pl}\\
&= \sum_{p=1}^{N}\left(q_{pk}-q_{p-1,k}\right)\left(q_{pl}-q_{p-1,l}\right)\sigma_p^2.\end{aligned}$$

□

Lemma 6.4.4 enables us to write the formula for $(R^{-1}AR^{-1})_{kl}$, where $A$ is given by (3.8), and $R$ is given by (3.16).

**Lemma 6.4.5.** *Suppose the matrix $R$ is given by* (3.16)*, and the matrix $A$ is given by* (3.8)*. Then for $1 \le k \le l \le N-1$ we have:*

$$\left(R^{-1}AR^{-1}\right)_{kl} = \frac{4(N-k)(N-l)}{N^2}\sum_{p=1}^{k}\sigma_p^2 - \frac{4k(N-l)}{N^2}\sum_{p=k+1}^{l}\sigma_p^2 + \frac{4kl}{N^2}\sum_{p=l+1}^{N}\sigma_p^2. \tag{6.35}$$

*Proof.* Apply Lemma 6.4.4 to $Q = R^{-1}$, given by (6.32), so that $q_{ij} = \rho_{ij}$. For $p \le k$, we get: For $p \le k$ we have:

$$\rho_{pk} - \rho_{p-1,k} = \frac{2p(N-k)}{N} - \frac{2(p-1)(N-k)}{N} = \frac{2(N-k)}{N},$$
$$\rho_{pl} - \rho_{p-1,l} = \frac{2p(N-l)}{N} - \frac{2(p-1)(N-l)}{N} = \frac{2(N-l)}{N}.$$

For $k < p \le l$, we have:

$$\rho_{pk} - \rho_{p-1,k} = \frac{2k(N-p)}{N} - \frac{2k(N-p+1)}{N} = -\frac{2k}{N},$$
$$\rho_{pl} - \rho_{p-1,l} = \frac{2p(N-l)}{N} - \frac{2(p-1)(N-l)}{N} = \frac{2(N-l)}{N}.$$

For $p > l$, we have:

$$\rho_{pk} - \rho_{p-1,k} = \frac{2p(N-k)}{N} - \frac{2(p-1)(N-k)}{N} = \frac{2(N-k)}{N},$$

$$\rho_{pl} - \rho_{p-1,l} = \frac{2p(N-l)}{N} - \frac{2(p-1)(N-l)}{N} = \frac{2(N-l)}{N}.$$

The rest of the proof is trivial. □

### 6.4.5 *Proof of Theorem* 6.2.1

Use Theorem 6.3.2 and Corollary 6.3.3 for matrices $R$ and $A$, given by (3.16) and (3.8) respectively. We have the following sufficient condition for avoiding total collisions:

$$\mathrm{tr}\big(R^{-1}A\big) - 2\max_{1\le k\le l\le N-1}\frac{(R^{-1}AR^{-1})_{kl}}{\rho_{kl}} \ge 0. \tag{6.36}$$

For $1 \le k \le l \le N-1$, denote

$$c_{k,l}(\sigma) = \mathrm{tr}\big(R^{-1}A\big) - 2\frac{\big(R^{-1}AR^{-1}\big)_{kl}}{\rho_{kl}}.$$

Then we have:

$$\mathrm{tr}\big(R^{-1}A\big) - 2\max_{k,l=1,\dots,N-1}\frac{(R^{-1}AR^{-1})_{kl}}{\rho_{kl}} = \min_{1\le k\le l\le N-1} c_{k,l}(\sigma). \tag{6.37}$$

**Lemma 6.4.6.** *Using definitions of $c_l(\sigma)$ and $\sigma^{\leftarrow}$ from subsection 1.2, we have:*

*(i) For $2 \le k \le l \le N-2$, we have: $c_{k,l}(\sigma) \ge 0$.*

*(ii) For $1 = k \le l \le N-1$, we have: $c_{k,l}(\sigma) = c_l(\sigma)$.*

*(iii) For $1 \le k \le l = N-1$, we have: $c_{k,l}(\sigma) = c_{N-k}\left(\sigma^{\leftarrow}\right)$.*

Assuming that Lemma 6.4.6 is proved, let us finish the proof of Theorem 6.2.1. Let

$$\delta(\sigma) := \min_{2\le k\le l\le N-2} c_{k,l}(\sigma). \tag{6.38}$$

If $N < 4$, let $\delta(\sigma) := 0$. By Lemma 6.4.6 (i), we always have: $\delta(\sigma) \ge 0$. Recall the definition of $\mathcal{P}(\sigma)$ from (6.1) and use Lemma 6.4.6 (ii), (iii):

$$\min\left(c_{1,1}(\sigma),\, c_{1,2}(\sigma),\, \dots,\, c_{1,N-1}(\sigma),\, c_{2,N-1}(\sigma),\, \dots,\, c_{N-1,N-1}(\sigma)\right) = \mathcal{P}(\sigma). \tag{6.39}$$

Comparing (6.37), (6.38) and (6.39), we have:

$$\min_{1\le k\le l\le N-1}\left[\mathrm{tr}\big(R^{-1}A\big) - 2\frac{(R^{-1}AR^{-1})_{kl}}{\rho_{kl}}\right] = \min(\mathcal{P}(\sigma), \delta(\sigma)). \tag{6.40}$$

Thus

$$\min_{1\le k\le l\le N-1} c_{k,l}(\sigma)\ge 0 \ \text{ if and only if } \ \mathcal{P}(\sigma)\ge 0.$$

This completes the proof of Theorem 6.2.1. $\square$

*Proof of Lemma* 6.4.6: We can simplify the expression for $c_{k,l}(\sigma)$. Applying (6.35) and (6.32), we have: for $1\le k\le l\le N-1$,

$$\frac{\left(R^{-1}AR^{-1}\right)_{kl}}{\rho_{kl}} = \frac{2(N-k)}{Nk}\sum_{p=1}^{k}\sigma_p^2 - \frac{2}{N}\sum_{p=k+1}^{l}\sigma_p^2 + \frac{2l}{N(N-l)}\sum_{p=l+1}^{N}\sigma_p^2.$$

Therefore, we have:

$$\begin{aligned}
c_{k,l}(\sigma) :=& \left(\frac{2(N-1)}{N} - \frac{4(N-k)}{Nk}\right)\sum_{p=1}^{k}\sigma_p^2 \\
&+ \left(\frac{2(N-1)}{N} + \frac{4}{N}\right)\sum_{p=k+1}^{l}\sigma_p^2 + \left(\frac{2(N-1)}{N} - \frac{4l}{(N-l)N}\right)\sum_{p=l+1}^{N}\sigma_p^2 \\
&= \frac{2(N-1)k-4(N-k)}{kN}\sum_{p=1}^{k}\sigma_p^2 + \frac{2(N+1)}{N}\sum_{p=k+1}^{l}\sigma_p^2 \\
&+ \frac{2(N-1)(N-l)-4l}{(N-l)N}\sum_{p=l+1}^{M}\sigma_p^2.
\end{aligned}$$

Now, for $k\ge 2$ we get:

$$2(N-1)k-4(N-k)\ge 4(N-1)-4N+8=4\ge 0.$$

Similarly, for $l\le N-2$ we get:

$$2(N-1)(N-l)-4l\ge 0.$$

This proves part (i) of Lemma 6.4.6. Parts (ii) and (iii) are now straightforward. $\square$

### *6.4.6 Proof of Theorem* 6.2.3

Fix a subset $I\subseteq\{1,\dots,N-1\}$ such that $J\subseteq I$. Take the matrices $R$ and $A$ given by (3.16) and (3.8). Essentially, we need to prove the following lemma:

**Lemma 6.4.7.** *If the subset $I$ satisfies Assumption (B), then the process $Z = (Z(t), t \geq 0) = \mathrm{SRBM}^{|I|}([R]_I, 0, [A]_I)$ a.s. does not hit the origin at any time $t > 0$.*

If we prove Lemma 6.4.7, then Theorem 6.2.3 will automatically follow from this lemma and Theorem 6.3.4. The rest of this subsection is devoted to the proof of Lemma 6.4.7.

Let us investigate the structure of the matrices $[R]_I^{-1}$ and $[A]_I^{-1}$. Split $I$ into disjoint non-adjacent discrete intervals: $I = I_1 \cup I_2 \cup \ldots \cup I_r$. Since the matrices $R$ and $A$ are tridiagonal, the matrices $[R]_I$ and $[A]_I$ have the following block-diagonal form:

$$[R]_I = \mathrm{diag}\left([R]_{I_1}, \ldots, [R]_{I_r}\right), \quad [A]_I = \mathrm{diag}\left([A]_{I_1}, \ldots, [A]_{I_r}\right).$$

So the processes

$$[Z]_{I_j} = ([Z(t)]_{I_j}, t \geq 0), \quad j = 1, \ldots, s, \tag{6.41}$$

are independent SRBMs:

$$[Z]_{I_j} = \mathrm{SRBM}^{|I_j|}\left([R]_{I_j}, 0, [A]_{I_j}\right).$$

And for any subset

$$I' = I_{i_1} \cup \ldots \cup I_{i_s}$$

the process

$$[Z]_{I'} = ([Z(t)]_{I'}, t \geq 0) = \mathrm{SRBM}^{|I'|}\left([R]_{I'}, 0, [A]_{I'}\right).$$

*Remark* 21. If for some choice of $I'$ this process a.s. does not hit the origin of $\mathbb{R}_+^{|I'|}$ at any time $t > 0$, then the original process $Z$ a.s. does not hit the origin at any time $t > 0$, because of independence of (6.41).

Now, let us state three lemmas.

**Lemma 6.4.8.** *If at least two of the discrete intervals $I_1, \ldots, I_r$ are singletons, then $Z$ a.s. at any time $t > 0$ does not hit the origin.*

**Lemma 6.4.9.** *If at least one $I_1, \ldots, I_r$ is a two-element subset $\{k-1, k\}$ with local concavity at $k$, then $Z$ a.s. at any time $t > 0$ does not hit the origin.*

**Lemma 6.4.10.** *If $I$ satisfies Assumption (A), then $Z$ a.s. at any time $t > 0$ does not hit the origin.*

Combining Lemmas 6.4.8, 6.4.9, and 6.4.10 with Remark 21, we complete the proof of Lemma 6.4.7 and Theorem 6.2.3. $\square$

In the remainder of this subsection, we shall prove these three lemmas.

*Proof of Lemma* 6.4.8: Without loss of generality, suppose $I_1 = \{k\}$ and $I_2 = \{l\}$ are singletons. Since they are not adjacent, $|k - l| \ge 2$; assume that $k < l$, so $l \ge k + 2$. Then

$$(Z_k, Z_l)' = \mathrm{SRBM}^2\left([R]_{I_1 \cup I_2}, 0, [A]_{I_1 \cup I_2}\right).$$

But

$$[A]_{I_1 \cup I_2} = \begin{bmatrix} \sigma_k^2 + \sigma_{k+1}^2 & 0 \\ 0 & \sigma_l^2 + \sigma_{l+1}^2 \end{bmatrix}, \quad [R]_{I_1 \cup I_2} = I_2.$$

So $Z_k$ and $Z_l$ are independent reflected Brownian motions on $\mathbb{R}_+$. They do not hit zero simultaneously, which is the same as to say that $(Z_k, Z_l)'$ does not hit the origin in $\mathbb{R}^2_+$.

*Proof of Lemma* 6.4.9: Follows from Remark 19 and Proposition 5.1.1.

*Proof of Lemma* 6.4.10: By Lemma 2.2.1 from Chapter 4, which corresponds to [100, Lemma 5.6], the matrices $[R]_{I_1}, \ldots, [R]_{I_r}$ are themselves reflection nonsingular $\mathcal{M}$-matrices, so they are invertible, and

$$[R]^{-1} = \mathrm{diag}\left([R]_{I_1}^{-1}, \ldots, [R]_{I_r}^{-1}\right).$$

In addition,

$$[R]_I^{-1}[A]_I^{-1} = \mathrm{diag}\left([R]_{I_1}^{-1}[A]_{I_1}, \ldots, [R]_{I_r}^{-1}[A]_{I_r}\right), \tag{6.42}$$

$$[R]_I^{-1}[A]_I^{-1}[R]_I^{-1} = \mathrm{diag}\left([R]_{I_1}^{-1}[A]_{I_1}[R]_{I_1}^{-1}, \ldots, [R]_{I_r}^{-1}[A]_{I_r}[R]_{I_r}^{-1}\right).$$

**Lemma 6.4.11.** *For the matrices $R$ and $A$ given by* (3.16) *and* (3.8)*, we have:*

$$\mathrm{tr}\left([R]_I^{-1}[A]_I^{-1}\right) = \sum_{j=1}^{r} \mathcal{T}(\overline{I}_j). \tag{6.43}$$

*Proof.* Because of (6.42), we get:

$$\operatorname{tr}\big([R]_I^{-1}[A]_I^{-1}\big) = \sum_{j=1}^{r} \operatorname{tr}\big([R]_{I_j}^{-1}[A]_{I_j}\big). \tag{6.44}$$

Applying Lemma 6.4.3 with $I_j$ instead of $\{1,\ldots,N-1\}$ and $\overline{I}_j$ instead of $\{1,\ldots,N\}$, $j = 1,\ldots,r$, we have:

$$\operatorname{tr}\big([R]_{I_j}^{-1}[A]_{I_j}\big) = \sum_{j=1}^{r} \mathcal{T}(\overline{I}_j), \quad j = 1,\ldots,r. \tag{6.45}$$

Combining (6.44) and (6.45), we get (6.43). □

**Lemma 6.4.12.** *We have the following estimate:*

$$\max_{x\in\mathbb{R}_+^{|I|}\setminus\{0\}} \frac{x'[R]_I^{-1}[A]_I[R]_I^{-1}x}{x'[R]_I^{-1}x} \le \max_{j=1,\ldots,r} \max_{\substack{k,l\in I_j\\ k\le l}} \frac{\big([R]_{I_j}^{-1}[A]_{I_j}[R]_{I_j}^{-1}\big)_{kl}}{\big([R]_{I_j}^{-1}\big)_{kl}}. \tag{6.46}$$

The proof of Lemma 6.4.12 is postponed until the end of this section. Assuming we have proved it, let us show how to finish the proof of Lemma 6.4.10.

Using (6.46) and (6.43), we can rewrite the condition (6.31) as

$$\sum_{j=1}^{r} \mathcal{T}(\overline{I}_j) - 2\max_{i=1,\ldots,r} \max_{\substack{k,l\in I_i\\ k\le l}} \frac{\big([R]_{I_i}^{-1}[A]_{I_i}^{-1}[R]_{I_i}^{-1}\big)_{kl}}{\big([R]_{I_i}^{-1}\big)_{kl}} \ge 0.$$

Equivalently,

$$\sum_{\substack{j=1\\ j\ne i}}^{r} \mathcal{T}(\overline{I}_j) + \mathcal{T}(\overline{I}_i) - 2\max_{\substack{k,l\in I_i\\ k\le l}} \frac{\big([R]_{I_i}^{-1}[A]_{I_i}^{-1}[R]_{I_i}^{-1}\big)_{kl}}{\big([R]_{I_i}^{-1}\big)_{kl}} \ge 0, \quad i = 1,\ldots,r.$$

In the proof of Theorem 6.2.1, see (6.40) and (6.34), it was shown that for $i = 1,\ldots,r$, we have:

$$\mathcal{T}(\overline{I}_i) - 2\max_{\substack{k,l\in I_i\\ k\le l}} \frac{\big([R]_{I_i}^{-1}[A]_{I_i}^{-1}[R]_{I_i}^{-1}\big)_{kl}}{\big([R]_{I_i}^{-1}\big)_{kl}} = \min(\mathcal{P}(\overline{I}_i), \delta_i), \quad \delta_i := \delta([\sigma]_{\overline{I}_i}) \ge 0.$$

Therefore, the condition (6.31) is equivalent to

$$\sum_{j\ne i} \mathcal{T}(\overline{I}_j) + \min(\mathcal{P}(\overline{I}_i), \delta_i) \ge 0, \quad i = 1,\ldots,r. \tag{6.47}$$

It suffices to note that $\mathcal{T}(\bar{I}_i) > 0$ for all $i$. Therefore, the condition (6.47), in turn, is equivalent to

$$\sum_{j \neq i} \mathcal{T}(\bar{I}_j) + \mathcal{P}(\bar{I}_i) \geq 0, \quad i = 1, \ldots, r.$$

This completes the proof of Lemma 6.4.10, and with it the proofs of Lemma 6.4.7 and Theorem 6.2.3.

*Proof of Lemma* 6.4.12. The matrices $[R]_I^{-1}$ and $[A]_I^{-1}$ are block-diagonal, with the blocks corresponding to the sets $I_1, \ldots, I_r$ of indices. Therefore,

$$x'[R]_I^{-1}[A]_I[R]_I^{-1}x = \sum_{j=1}^{r}[x]'_{I_j}[R]_{I_j}^{-1}[A]_{I_j}[R]_{I_j}^{-1}[x]_{I_j}, \quad x'[R]_I^{-1}x = \sum_{j=1}^{r}[x]'_{I_j}[R]_{I_j}^{-1}[x]_{I_j}. \tag{6.48}$$

Let $\mathcal{Q}(x) := \{j = 1, \ldots, r \mid [x]_{I_j} \neq 0\}$. We might as well rewrite (6.48) as

$$x'[R]_I^{-1}[A]_I[R]_I^{-1}x = \sum_{j \in \mathcal{Q}(x)} [x]'_{I_j}[R]_{I_j}^{-1}[A]_{I_j}[R]_{I_j}^{-1}[x]_{I_j}, \quad x'[R]_I^{-1}x = \sum_{j \in \mathcal{Q}(x)} [x]'_{I_j}[R]_{I_j}^{-1}[x]_{I_j}.$$

For $j \in \mathcal{Q}(x)$, we have: $[x]_{I_j} \in S_+^{|I_j|} \setminus \{0\}$. The matrix $[R]_{I_j}$ has the same form as $R$ in (3.16), but with smaller size. So all elements of the inverse matrix $[R]_{I_i}^{-1}$ (just like for $R^{-1}$) are positive. Therefore, $[x]'_{I_i}[R]_{I_i}^{-1}[x]_{I_i} > 0$, $i = 1, \ldots, r$. Applying Lemma 6.7.1 to $a_i = [x]'_{I_i}[R]_{I_i}^{-1}[A]_{I_i}[R]_{I_i}^{-1}[x]_{I_i}$ and $b_i = [x]'_{I_i}[R]_{I_i}^{-1}[x]_{I_i} > 0$ for $i \in \mathcal{Q}(x)$, we get:

$$\frac{x'[R]_I^{-1}[A]_I[R]_I^{-1}x}{x'[R]_I^{-1}x} \leq \max_{j \in \mathcal{Q}(x)} \frac{[x]'_{I_j}[R]_{I_j}^{-1}[A]_{I_j}[R]_{I_j}^{-1}[x]_{I_j}}{[x]'_{I_j}[R]_{I_j}^{-1}[x]_{I_j}}. \tag{6.49}$$

But the matrix $[R]_{I_j}$, as just mentioned, has all elements positive. Applying Lemma 6.3.3, we have for $y \in \mathbb{R}_+^{|I_j|} \setminus \{0\}$:

$$\frac{y'[R]_{I_j}^{-1}[A]_{I_j}[R]_{I_j}^{-1}y}{y'[R]_{I_j}^{-1}y} \leq \max_{\substack{k,l \in I_j \\ k \leq l}} \frac{\left([R]_{I_j}^{-1}[A]_{I_j}[R]_{I_j}^{-1}\right)_{kl}}{\left([R]_{I_j}^{-1}\right)_{kl}}. \tag{6.50}$$

Combining (6.49) and (6.50), we get (6.46). □

## 6.5 *Proof of Theorem 6.2.2*

Recall the setting of Theorem 6.3.2: we have a process $Z = (Z(t), t \geq 0)$ in $\mathbb{R}^d_+$, which is an $\mathrm{SRBM}^d(R, \mu, A)$ with a reflection nonsingular $\mathcal{M}$-matrix $R$. We would like this process to avoid the corner $\{0\}$.

A careful reading of the proof of Theorem 6.3.2 shows the following: Replace the matrix $R^{-1}$ in the formula for $F(Z(t))$ by a matrix $Q$. To guarantee that the process $F(Z(t)) = Z(t)'QZ(t)$ avoids the origin, we need the following conditions to hold:

$$z'Qz > 0, \quad \text{for all} \quad z \in \mathbb{R}^d_+; \tag{6.51}$$

$$(QR)_{i,j} \geq 0, \quad \text{if} \quad i \neq j; \tag{6.52}$$

$$\operatorname{tr}(QA) \geq 2 \max_{x \in \mathbb{R}^d_+ \setminus \{0\}} \frac{x'QAQx}{x'Qx}. \tag{6.53}$$

The first condition guarantees that $F(Z(t)) = 0$ is equivalent to $Z(t) = 0$. The third condition is needed to get the correct drift when comparing the time-changed process $F(Z(\cdot))$ to a Bessel process. The only change being made to this proof is (6.52). This condition guarantees that the terms corresponding to the regulating processes for the process $F(Z(\cdot))$ are all nonnegative. In the proof of Theorem 6.3.2, the matrix $Q = R^{-1}$ is chosen to make the inequality be an equality (and so the regulating processes go away). But if they are nondecreasing, this can only help the process $F(Z(\cdot))$ avoid hitting zero. So $Q = R^{-1}$ is only one of many possible choices. And here, we make a different choice for the matrix $Q$. Remember that (subsection 4.2) in this particular case the SRBM $Z = (Z(t), t \geq 0)$ is actually the gap process for the system of $N = 4$ competing Brownian particles. So we have: $d = N - 1 = 3$, and

$$R = \begin{bmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} \sigma_1^2 + \sigma_2^2 & -\sigma_2^2 & 0 \\ -\sigma_2^2 & \sigma_2^2 + \sigma_3^2 & -\sigma_3^2 \\ 0 & -\sigma_3^2 & \sigma_3^2 + \sigma_4^2 \end{bmatrix}$$

We pick the following matrix:

$$Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \text{where} \quad \lambda = \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{\sigma_2^2 + \sigma_3^2}. \tag{6.54}$$

Since every entry in $Q$ is positive, the condition (6.51) is easily satisfied. One can also confirm the relation $QAQ = \frac{tr(QA)}{2}Q$, and so (6.53) is satisfied. Finally, calculations show that

$$QR = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 1 - \frac{\lambda}{2} & \lambda - 1 & 1 - \frac{\lambda}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \tag{6.55}$$

and so (6.52) is equivalent to

$$1 - \frac{\lambda}{2} \ge 0 \iff \lambda \le 2 \iff \sigma_2^2 + \sigma_3^2 \ge \sigma_1^2 + \sigma_4^2.$$

## 6.6 The Case of Asymmetric Collisions

We can define collisions and multicollisions similarly to the classical case, as in Definition 29. It was shown in [71], see also Chapter 3, that the gap process for systems with asymmetric collisions, much like for the classical case, is an SRBM. Namely, it is an $\mathrm{SRBM}^{N-1}(R, \mu, A)$, where $\mu$ and $A$ are given by (3.9) and (3.8), and the reflection matrix $R$ is given by

$$R = \begin{bmatrix} 1 & -q_2^- & 0 & 0 & \ldots & 0 & 0 \\ -q_2^+ & 1 & -q_3^- & 0 & \ldots & 0 & 0 \\ 0 & -q_3^+ & 1 & -q_4^- & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & -q_{N-1}^- \\ 0 & 0 & 0 & 0 & \ldots & -q_{N-1}^+ & 1 \end{bmatrix} \tag{6.56}$$

The connection between multicollisions and multiple collisions in this system and hitting of edges of $\mathbb{R}_+^{N-1}$ by the gap process is the same as in Lemma 6.4.1. This allows us to apply

Theorem (6.3.2) and Theorem (6.3.4) to find sufficient conditions for avoiding multicollisions of a given pattern. In particular, the results of Lemma 6.4.2 are still valid for system with asymmetric collisions: the property of a.s. avoiding multicollisions of a certain pattern depends only on the diffusion coefficients and parameters of collision.

A remark is in order: the matrix $R$ in (6.56) in general is not symmetric, as opposed to the matrix $R$ in (3.16). But if we take the $(N-1)\times(N-1)$ diagonal matrix

$$C = \mathrm{diag}\left(1, \frac{q_2^+}{q_2^-}, \frac{q_2^+q_3^+}{q_2^-q_3^-}, \ldots, \frac{q_2^+q_3^+\ldots q_{N-1}^+}{q_2^-q_3^-\ldots q_{N-1}^-}\right),$$

then the matrix $\overline{R} = RC$ is diagonal.

## 6.7 *Appendix*

### *6.7.1 Proof of Lemma* 6.3.1

By [103, Lemma 2.1], the matrix $R^{-1}$ has all elements nonnegative, and its diagonal elements are strictly positive. The same is true for the matrix $R^{-1} = C^{-1}R^{-1} = (\rho_{ij})_{1\le i,j\le d}$. If $x \in S\setminus\{0\}$, then all components of $x$ are nonnegative with at least one component strictly positive, and so

$$x'R^{-1}x = \sum_{i=1}^d\sum_{j=1}^d \rho_{ij}x_ix_j \ge \sum_{i=1}^d \rho_{ii}x_i^2 > 0.$$

In addition, $R^{-1}$ is a nonsingular matrix, and $x \ne 0$, so $R^{-1}x \ne 0$. Since $A$ is positive definite, we have:

$$x'R^{-1}AR^{-1}x = (R^{-1}x)'A(R^{-1}x) > 0.$$

Therefore, the function

$$f(x) := \frac{x'R^{-1}AR^{-1}x}{x'R^{-1}x}$$

is well-defined and strictly positive on $S\setminus\{0\}$. In addition, it is *homogeneous*, in the sense that for $x \in S\setminus\{0\}$ and $k > 0$ we have: $f(kx) = f(x)$. Therefore,

$$\{f(x) \mid x \in S\setminus\{0\}\} = \{f(x) \mid x \in S,\ \|x\| = 1\}.$$

The set $\{x \in S \mid \|x\| = 1\}$ is compact, and $f$ is continuous and positive on this set. Therefore, it is bounded on this set (and so on $S \setminus \{0\}$), and reaches its maximal and minimal values, both of which are strictly positive.

### *6.7.2 Proof of Lemma* 6.3.3

Let us prove the statement for the maximum. For the minimum, the proof is similar. For $x \in S \setminus \{0\}$, we have: $x_1, \ldots, x_d \geq 0$, and so

$$\frac{x' R^{-1} A R^{-1} x}{x' R^{-1} x} = \frac{\sum_{i=1}^d \sum_{j=1}^d (R^{-1} A R^{-1})_{ij} x_i x_j}{\sum_{i=1}^d \sum_{j=1}^d \rho_{ij} x_i x_j}.$$

Apply Lemma 6.7.1 to $s = d^2$, $a_{ij} = (R^{-1} A R^{-1})_{ij} x_i x_j$, $b_{ij} = \rho_{ij} x_i x_j$ (we index $a_i$ and $b_i$ by double indices, with each of the two indices ranging from 1 to $d$). It suffices to note that, because of the symmetry of $R^{-1} A R^{-1}$ and $R^{-1} = (\rho_{ij})$, we have:

$$\max_{i,j=1,\ldots,d} \frac{(R^{-1} A R^{-1})_{ij}}{\rho_{ij}} = \max_{1 \leq i \leq j \leq d} \frac{(R^{-1} A R^{-1})_{ij}}{\rho_{ij}}.$$

### *6.7.3 A technical lemma*

**Lemma 6.7.1.** *Take real numbers $a_1, \ldots, a_s$ and positive real numbers $b_1, \ldots, b_s$. Then*

$$\min\left(\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\right) \leq \frac{a_1 + \ldots + a_s}{b_1 + \ldots + b_s} \leq \max\left(\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\right).$$

*Proof.* Let us prove the inequality

$$\frac{a_1 + \ldots + a_s}{b_1 + \ldots + b_s} \leq \max\left(\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\right).$$

The other inequality is proved similarly. Assume the converse: that

$$\frac{a_1 + \ldots + a_s}{b_1 + \ldots + b_s} > \frac{a_i}{b_i}, \quad i = 1, \ldots, s.$$

Multiply the $i$th inequality by $(b_1 + \ldots + b_s) b_i > 0$: $(a_1 + \ldots + a_s) b_i > a_i (b_1 + \ldots + b_s)$ for $i = 1, \ldots, s$. Add them up and arrive at a contradiction. $\square$

# Chapter 7

# INFINITE SYSTEMS

## *7.1 Introduction*

In this chapter, which corresponds to the author's paper [101], we prove Theorem 1.4.1 for the infinite Atlas model, together with similar results for more general infinite systems. In these systems, the $k$th ranked particle moves as a Brownian motion with drift $g_k$ and diffusion $\sigma_k^2$ (where $g_k, \sigma_k^2$ are fixed parameters) for each $k = 1, 2, \ldots$, see Definition 20. These are called *infinite classical systems of competing Brownian particles*, see [105], [59].

We devote Section 7.2 to the questions of existence and uniqueness results for these infinite classical systems, see Proposition 7.2.1. In fact, these statements were already proved in [105] and [59], but we provide a full proof here for the sake of completeness. The infinite Atlas model is a particular case of this general model, when

$$g_1 = 1,\ g_2 = g_3 = \ldots = 0,\ \sigma_1 = \sigma_2 = \ldots = 1.$$

In Section 7.3, Definition 32, we introduce a generalization of such systems: infinite systems with *asymmetric collisions*, with *parameters of collisions* $(q^{\pm}_k)_{k\ge1}$. This means, so to speak, that ranked particles $Y_k,\ k = 1, 2, \ldots$, have "different mass", and when they collide, they fly apart with "different speed". Finite systems with asymmetric collisions were already defined in [71], see also Chapter 3 of the current thesis.

Section 7.4 is devoted to the gap process $Z = (Z(t), t \ge 0)$: stationary distributions and weak convergence as $t \to \infty$. In subsection 7.4.1, Theorem 7.4.3, we construct a stationary distribution $\pi$ for the gap process of an infinite system, as a limit of stationary distributions for finite systems. For the infinite Atlas model, this distribution $\pi$ is none other than $\pi_\infty$. By the way, this provides a simplified proof of the main result of [89]: that $\pi_\infty$ is a stationary distribution for the gap process of the infinite Atlas model. But we do this for the general case of an infinite system with asymmetric collisions. In subsection 7.4.2, we elaborate on results of subsection 7.4.1. We consider the case when this stationary distribution $\pi$ is of product-of-exponentials form (the so-called *skew-symmetry condition*).

In subsection 7.4.3, we consider weak convergence of the gap process $Z(t)$ as $t \to \infty$. We prove Theorem 7.4.5, which is essentially the same as Theorem 1.4.1, but for the general case instead of the infinite Atlas model. Theorem 1.4.1 turns out to be a straightforward corollary of Theorem 7.4.5.

In Section 7.5, we prove results about a.s. absence of triple and simultaneous collisions, continuing the work done in Chapter 5 (which corresponds to the author's paper [103]). The results for infinite systems turn out to be very similar to that for finite systems. Section 7.6 is devoted to proofs of some results from the paper. Section 7.7 (Appendix) contains a statement and a proof of some technical lemmata.

## 7.2 Infinite Classical Systems of Competing Brownian Particles

The following existence and uniqueness theorem was proved in [59] and [105]. We restate it here in a slightly different form and prove, combining the proofs from these two articles. For the sake of completeness, we include the whole proof in this thesis, although in the article [101] we have only part of this proof.

**Proposition 7.2.1.** *Suppose $x \in \mathbb{R}^\infty$ is a rankable vector which satisfies the following condition:*

$$\lim_{n\to\infty} x_n = \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} e^{-\alpha x_n^2} < \infty \quad \text{for all} \quad \alpha > 0. \tag{7.1}$$

*Assume also that there exists $n_0 \geq 1$ for which*

$$g_{n_0+1} = g_{n_0+2} = \ldots = g, \quad \text{and} \quad \sigma_{n_0+1} = \sigma_{n_0+2} = \ldots = \sigma > 0.$$

*Then, in a weak sense there exists an infinite classical system of competing Brownian particles with drift coefficients $(g_k)_{k\geq 1}$ and diffusion coefficients $(\sigma_k^2)_{k\geq 1}$, starting from $x$, and it is unique in law.*

*Proof.* Without loss of generality, assume $x_1 \le x_2 \le \ldots$ Construction of this system goes as follows: for every $N \ge n_0$ and $x \in \mathbb{R}^N$, take a probability space

$$\left(\Omega^{(N,x)}, \mathcal{F}^{(N,x)}, \mathbf{P}^{(N,x)}\right)$$

with a classical system

$$X^{(N,x)} = \left(X_1^{(N,x)}, \ldots, X_N^{(N,x)}\right)'$$

of $N$ competing Brownian particles, with drift coefficients $(g_n)_{1 \le n \le N}$ and diffusion coefficients $(\sigma_n^2)_{1 \le n \le N}$, starting from $X^{(N,x)}(0) = x$, and with a sequence $B_1^{(N,x)}, B_2^{(N,x)}, \ldots$ of i.i.d. standard Brownian motions, independent of the system $X^{(N,x)}$. Now, consider the product $(\Omega, \mathcal{F}, \mathbf{P})$ of all these probability spaces.

Define the infinite system $X$ recursively. First,

$$N_0 := n_0,\ \tau_0 := 0,\ X(0) = x.$$

Next, for every $m = 0, 1, \ldots$, as $t \le \tau_{m+1} - \tau_m$, we define: $x_m := (X_1(\tau_m), \ldots, X_{N_m}(\tau_m))'$, and

$$X_i\left(t + \tau_m\right) = \begin{cases} X_i(\tau_m) + gt + \sigma W_i(t), & i > N_m; \\ X_i^{(N_m, x_m)}(t),\ i = 1, \ldots, N_m. \end{cases}$$

Here,

$$\tau_{m+1} := \inf\{t \ge \tau_m \mid \exists\, i > N_m,\ j = 1, \ldots, n_0 : \quad X_i(t) = X_j(t)\},$$

and

$$N_{m+1} := \max\{i > N_m \mid \exists\, j = 1, \ldots, n_0 :\ X_i\left(\tau_{m+1}\right) = X_j\left(\tau_{m+1}\right)\}.$$

Let us explain the method of construction in words. All particles, except the lowest $n_0$ ranked ones, move as Brownian motions with drift $g$ and diffusion $\sigma^2$. Initially, we define the infinite system by splitting it into two parts. The first part is the bottom $n_0$ particles (which coincide with the particles $X_1, \ldots, X_{n_0}$, because at time $t = 0$, ranks coincide with names). They move as a finite classical system of $n_0$ competing Brownian particles. The second part of this infinite system consists of countably many independent Brownian motions, starting from $x_i,\ i > n_0$, each having drift $g$ and diffusion $\sigma^2$.

The particles follow this dynamics until some particle from the second part collides with some particle with the first part. Let us call this moment $\tau_1$. Then we add this particle (and all other particles from the second part hit by some particles from the first part, if there is more than one collision at this moment $\tau_1$) to the first part, which becomes bigger. Let $N_1$ be the largest name of a particle in the updated first part. Then we add all particles with names less than or equal to $N_1$ to the first part of the infinite system, even if they have not hit one of the particles $X_1, \ldots, X_{n_0}$.

Starting from the moment $\tau_1$, we define the updated first part to be a finite classical system of competing Brownian particles, and the updated second part to be again just independent Brownian motions. Particles follow this dynamics until there is another collision between two particles: one from the first part and one from the second part. We call this moment $\tau_2$ and update the parts of the system again. Then we repeat the process.

Suppose that we have proved the following statements.

**Lemma 7.2.2.** *For every $m = 1, 2, \ldots$ we have: $N_m < \infty$ a.s.*

**Lemma 7.2.3.** *As $m \to \infty$, we have: $\tau_m \to \infty$ a.s.*

Let us show that the process $X$ is, in fact, an infinite classical system of competing Brownian particles with drift coefficients $(g_n)_{n\ge1}$ and diffusion coefficients $(\sigma_n^2)_{n\ge1}$, starting from $X(0) = x$.

One can describe the behavior of the infinite classical system as consisting of the bottom $n_0$ particles, which have drift coefficients $(g_n)_{1\le n\le n_0}$ and diffusion coefficients $(\sigma_n^2)_{1\le n\le n_0}$, and all other particles, which have drift coefficient $g$ and diffusion coefficient $\sigma^2$. As long as a particle from the second group does not hit a particle from the first group, it behaves as a Brownian motion with drift $g$ and diffusion $\sigma^2$, without interacting with other particles from the second group. As it hits one of those “exceptional ” bottom $n_0$ particles, however, it needs to be “integrated” into a finite system of competing Brownian particles. We can carry out the construction so that this "integration" occurs before this moment, but not after.

It suffices to show that this is the case for our construction: that each of the "upper" particles is "integrated" before or at least at the moment of hitting one of the lowest-ranked $n_0$ particles. Indeed, assume

$$X_i(t) = X_{\mathbf{p}_t(j)}(t) \text{ for some } i > n_0,\ j = 1, \dots, n_0.$$

We claim that then there exists $s \in [0, t]$ and $l = 1, \dots, j$ such that $X_i(s) = X_l(s)$. Indeed, the quantity of $k$ such that $X_k(t) < X_i(t)$ is less than $j$ (this follows from the definition of rank). So there exists $l = 1, \dots, j$ such that $X_l(0) \le X_i(0)$, but $X_l(t) > X_i(t)$. Using continuity of $X_l$ and $X_i$, we complete the proof.

Now, let us show uniqueness in law of the system $X$: it holds until every moment $\tau_m$, and since $\tau_m \to \infty$, it holds on the whole infinite time horizon.

*Proof of Lemma* 7.2.2. Assume the converse: that $N_j = \infty$ for some $j$. Denote this event by $A_\infty$. Since the sequence $(N_m)_{m\ge0}$ is strictly increasing, there exists $m$ such that $N_m < \infty$, but $N_{m+1} = \infty$. Therefore,

$$A_\infty = \bigcup_{m=0}^{\infty}\bigcup_{M=0}^{\infty} A(M, m), \quad \text{where } \ A(M, m) := \{N_m = M,\ N_{m+1} = \infty\}. \tag{7.2}$$

Assume the event $A(M, m)$ has happened. Then $\tau_{m+1} < \infty$. The fact that $N_{m+1} = \infty$ means that $X_i(\tau_{m+1})$ is the same for infinitely many values of $i$; in particular, for infinitely many values of $i > n_0$. But for $i > M = N_m$, the processes $X_i$ behave as Brownian motions with drift $g$ and diffusion $\sigma^2$, starting from $X_i(0) = x_i$, at least until $\tau_{m+1}$. Among these Brownian motions, there exist three: $X_{i_1}, X_{i_2}, X_{i_3}$, which start from different $x_i$ (because $x_i \to \infty$ as $i \to \infty$). Therefore, the event

$$A(M, m, i_1, i_2, i_3) := A(M, m) \cap \{X_{i_1}(\tau_{m+1}) = X_{i_2}(\tau_{m+1}) = X_{i_3}(\tau_{m+1})\}$$

has probability zero. But

$$A(M, m) = \bigcup_{i_1, i_2, i_3} A(M, m, i_1, i_2, i_3),$$

where the union is taken over all $i_1, i_2, i_3 > M$ such that $x_{i_1}, x_{i_2}, x_{i_3}$ are all different. This is a countable union, so $\mathbf{P}(A(M, m)) = 0$. Thus, from (7.2) we have: $\mathbf{P}(A_\infty) = 0$.

*Proof of Lemma* 7.2.3. Fix $T > 0$. It suffices to show that

$$\lim_{m\to\infty} \mathbf{P}(\tau_m \le T) = 0.$$

Fix $m \ge 1$ and assume the event $\{\tau_m \le T\}$ has happened. Note that, until the moment $\tau_m$, the first $N_m$ components $(X_1, \ldots, X_{N_m})'$ of the system $X$ behave as a finite classical system

$$\tilde{X}(t) = \left(\tilde{X}_1(t), \ldots, \tilde{X}_{N_m}(t)\right)'$$

of $N_m$ competing Brownian particles:

$$\left((X_1(t), \ldots, X_{N_m}(t))',\ 0 \le t \le \tau_m\right) = \left(\left(\tilde{X}_1(t), \ldots, \tilde{X}_{N_m}(t)\right)',\ 0 \le t \le \tau_m\right).$$

Therefore,

$$\max_{0\le t\le \tau_m} \max_{1\le i\le n_0} X_i(t) \le \max_{0\le t\le T} \max_{1\le i\le n_0} \tilde{X}_i(t) < \infty.$$

Fix $\varepsilon > 0$. Take a threshold $u_\varepsilon \in \mathbb{R}$ such that

$$\mathbf{P}\left(\max_{0\le t\le T} \max_{1\le i\le n_0} \tilde{X}_i(t) > u_\varepsilon\right) < \varepsilon.$$

Note that this $u_\varepsilon$ does not depend on $m$. Now, each $X_i$ for $i > N_m$ behaves as a Brownian motion with drift $g$ and diffusion $\sigma^2$, starting from $x_i$, at least until the moment $\tau_m$. Therefore, for some standard Brownian motion $\tilde{B}_i$ we have:

$$X_i(t) = x_i + gt + \sigma \tilde{B}_i(t), \quad t \le \tau_m.$$

Suppose we proved that

$$\sum_{i>N_m}^{\infty} \mathbf{P}\left(\min_{0\le t\le T} X_i(t) < u_\varepsilon\right) < \infty. \tag{7.3}$$

Applying the Borel-Cantelli lemma, we get: there are a.s. only finitely many $i > N_m$ such that

$$\min_{0\le t\le \tau_m} X_i(t) < u_\varepsilon.$$

There exists $M_\varepsilon$ such that the number of these $i \ge 1$ is greater than $M_\varepsilon$ only with probability $\le \varepsilon$. And this $M_\varepsilon$ is independent of $m$: it depends only on $T$ and $\varepsilon$. So with probability $\ge 1 - 2\varepsilon$, there does *not* exist $i > M_\varepsilon$ and $j = 1, \ldots, n_0$ such that

$$X_i(\tau_m) = X_j(\tau_m).$$

If this event happened, then $N_m \le M_\varepsilon$. From construction of $(N_j)_{j\ge 0}$ we know that $N_j \ge N_{j-1} + 1$ for all $j$, and so $N_m \ge N_0 + m = n_0 + m$. Therefore,

$$\mathbf{P}(m \le M_\varepsilon - n_0) > 1 - 2\varepsilon.$$

If we fix $\varepsilon > 0$ and take $m_\varepsilon := M_\varepsilon - n_0 + 1$, then

$$\mathbf{P}\left(\tau_{m_\varepsilon} > T\right) > 1 - 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily small, this completes the proof.

Now, let us show (7.3). Indeed,

$$\left\{\min_{0\le t\le T} X_i(t) < u_\varepsilon\right\} \subseteq \left\{x_i - (gT)_- + \sigma \min_{0\le t\le T} \tilde{B}_i(t) < u_\varepsilon\right\} \subseteq \left\{\sigma \min_{0\le t\le T} \tilde{B}_i(t) < u_\varepsilon - x_i + (gT)_-\right\},$$

and the sum in (7.3) can be estimated as

$$\sum_{i>N_m}^{\infty} \mathbf{P}\left(\sigma \min_{0\le t\le T} \tilde{B}_i(t) < u_\varepsilon - x_i + (gT)_-\right) = \sum_{i>N_m}^{\infty} 2\mathbf{P}\left(\sigma \tilde{B}_i(T) > x_i - u_\varepsilon - (gT)_-\right)$$
$$= 2\sum_{i>N_m}^{\infty} \Psi\left(\frac{x_i - u_\varepsilon - (gT)_-}{\sigma\sqrt{T}}\right).$$

Using Lemma 7.7.1, we conclude that the latter sum is finite. $\square$

Let us state a few properties of this infinite classical system of competing Brownian particles. They were already stated and proved in [105] and [59], but we include the proof for the sake of completeness.

**Proposition 7.2.4.** *Under conditions of Proposition* 7.2.1*, we have:*

*(i) For every $T > 0$ and $u \in \mathbb{R}$ there are a.s. only finitely many particles $X_i$ such that*

$$\min_{0 \le t \le T} X_i(t) < u.$$

*(ii) Moreover, for every $\alpha > 0$ we have:*

$$\sum_{i=1}^{\infty} e^{-\alpha X_i^2(t)} < \infty.$$

*(iii) The dynamics of the ranked particles $Y_k$ is as follows. Denote by*

$$L_{(k,k+1)} = (L_{(k,k+1)}(t), t \ge 0)$$

*the local time process at zero of $Z_k$, $k = 1, 2, \ldots$. For notational convenience, let $L_{(0,1)}(t) \equiv 0$. Let*

$$B_k(t) = \sum_{i=1}^{\infty} \int_0^t 1(\mathbf{p}_s(k) = i) \mathrm{d}W_i(s), \quad k = 1, 2, \ldots, \quad t \ge 0.$$

*Then the processes $B_k = (B_k(t), t \ge 0)$, $k = 1, 2, \ldots$ are i.i.d. standard Brownian motions. We have:*

$$Y_k(t) = Y_k(0) + g_k t + \sigma_k B_k(t) - \frac{1}{2} L_{(k,k+1)}(t) + \frac{1}{2} L_{(k-1,k)}(t), \ t \ge 0, \ k = 1, 2, \ldots. \tag{7.4}$$

*Proof.* (i) We can write $X_i(t)$ in the form of

$$X_i(t) = y_i + \int_0^t \beta_i(s) \mathrm{d}s + \int_0^t \rho_i(s) \mathrm{d}s, \tag{7.5}$$

where

$$\beta_i(t) := \sum_{k=1}^{\infty} 1(X_i \ \text{ has rank } \ k \ \text{ at time } \ t) g_k, \quad \rho_i(t) := \sum_{k=1}^{\infty} 1(X_i \ \text{ has rank } \ k \ \text{ at time } \ t) \sigma_k.$$

Because of (7.8), we get:

$$|\beta_i(t)| \le \overline{g}, \quad |\rho_i(t)| \le \overline{\sigma}, \quad t \ge 0.$$

Therefore,

$$X_i(t) \ge y_i - \overline{g} T + \mathcal{M}_i(t), \quad \text{where} \ \ \mathcal{M}_i(t) := \int_0^t \rho_i(s) \mathrm{d}W_i(s)$$

is a continuous square-integrable martingale with quadratic variation

$$\langle \mathcal{M}_i \rangle_t = \int_0^t \rho_i^2(s) \mathrm{d}s \le \overline{s}^2 T, \quad t \in [0, T].$$

Let us make a time-change: for some standard Brownian motion $\overline{B}_i = (\overline{B}_i(s), s \ge 0)$, we have:

$$\mathcal{M}_i(t) = \overline{B}_i \left( \langle \mathcal{M}_i \rangle_t \right).$$

So

$$\min_{0 \le t \le T} X_i(t) \ge y_i - (\overline{g}T)_+ + \min_{[0, \overline{\sigma}^2 T]} \overline{B}_i(s).$$

Therefore,

$$\begin{aligned}
\mathbf{P}\left( \min_{0 \le t \le T} X_i(t) < u \right) &\le \mathbf{P}\left( \min_{0 \le t \le \overline{\sigma}^2 T} \overline{B}_i(t) < u - y_i + (\overline{g}T)_+ \right) \\
&= \mathbf{P}\left( \max_{0 \le t \le \overline{\sigma}^2 T} \overline{B}_i(t) > -u + y_i - (\overline{g}T)_+ \right) = 2\mathbf{P}\left( \overline{B}_i(\overline{\sigma}^2 T) > -u + y_i - (\overline{g}T)_+ \right) \\
&= 2\Psi\left( \frac{-u + y_i - (\overline{g}T)_+}{\overline{\sigma}\sqrt{T}} \right).
\end{aligned}$$

By Lemma 7.7.1, the sum of the terms on the right is finite. By the Borel-Cantelli lemma, (i) is proved.

(ii) We use the representation (7.5). Let us show that

$$\sum_{i=1}^{\infty} \mathbf{P}\left( \min_{0 \le t \le \overline{\sigma}^2 T} \overline{B}_i(t) < -\frac{1}{2} y_i \right) < \infty. \tag{7.6}$$

Indeed,

$$\mathbf{P}\left( \min_{0 \le t \le \overline{\sigma}^2 T} \overline{B}_i(t) < -\frac{1}{2} y_i \right) = \mathbf{P}\left( \max_{0 \le t \le \overline{\sigma}^2 T} \overline{B}_i(t) > \frac{1}{2} y_i \right) = \Psi\left( \frac{y_i/2}{\overline{\sigma}\sqrt{T}} \right).$$

Then it suffices to apply Lemma 7.7.1. Now, (7.6) means that (applying the Borel-Cantelli lemma), for all but finitely many $i \ge 1$ such that

$$\min_{0 \le t \le \overline{\sigma}^2 T} \overline{B}_i(t) \ge -\frac{1}{2} y_i.$$

Therefore, for these (all but finitely many) $i \ge 1$

$$y_i + (\overline{g}T)_- + \min_{0 \le t \le \overline{\sigma}^2 T} \overline{B}_i(t) \ge \frac{1}{2} y_i + (\overline{g}T)_+. \tag{7.7}$$

By Lemma 7.7.1, for all $\alpha > 0$,

$$\sum_{i=i_0}^{\infty} e^{-\alpha((1/2)y_i+(\overline{g}T)_+)^2} < \infty.$$

This completes the proof of (ii).

(iii) This statement follows from (i) and similar statement for finite systems (see (3.2)). Indeed, take the $k$th ranked particle $Y_k$ and let $u := \max_{[0,T]} Y_k + 1$. Let us show that for every $t \in [0,T]$ there exists a neighborhood of $t$ in $[0,T]$ (possibly random) such that (7.4) holds. The statement of (iii) would then follow from compactness of $[0,T]$ and the fact that $T > 0$ is arbitrary.

Indeed, there exists $i_0$ such that $\min_{[0,T]} X_i > u$ for $i > i_0$. Take $m > k$ and assume the event $\{i_0 \le m\}$ happened. Fix time $t \in [0,T]$. We claim that if $Y_k$ does not collide at time $t$ with other particles, then there exists a (random) neighborhood when $Y_k$ does not collide with other particles. Indeed, particles $X_i,\ i > m$, cannot collide with $Y_k$, by definition of $u$ and $i_0$. And for every particle $X_i,\ i = 1,\dots,m$, other than $Y_k$ (say $Y_k$ has name $j$ at time $t$), there exists an open neighborhood of $t$ such that this particle does not collide with $Y_k = X_j$ in this neighborhood. Take the finite intersection of these $m-1$ neighborhoods and complete the proof of the claim.

In this case, the formula 7.4 is trivial, because the local time terms $L_{(k-1,k)}$ and $L_{(k,k+1)}$ are constant in this neighborhood.

Now, if $Y_k(t)$ does collide with particles $X_i,\ i \in I$, then $I \subseteq \{1,\dots,m\}$. We claim that there exists a neighborhood of $t$ such that, in this neighborhood, particles $X_i,\ i \in I$, do not collide with any other particles. Indeed, for every $i \in I$, we have: $X_i(t) = Y_k(t) \le u - 1$. There exists a neighborhood of $t$ in which $X_i$ does not collide with any particles $X_l,\ l \in \{1,\dots,m\} \setminus I$. There exists another neighborhood in which $X_i(t) < u$ (and therefore it does not collide with any particles $X_l,\ l > m$). Intersect all these neighborhoods (there are $2|I|$ of them) and complete the proof of this claim.

In this neighborhood, the system $(X_i, i \in I)$ behaves as a finite system of competing Brownian particles. It suffices to refer to (3.2). $\square$

## 7.3 Infinite Systems with Asymmetric Collisions

Proposition 7.2.4 provides motivation to introduce infinite systems of competing Brownian particles with *asymmetric collisions*, when we have coefficients other than $1/2$ at the local times in (7.4). We prove an existence theorem for these systems. Unfortunately, we could not prove uniqueness: we just construct a copy of an infinite ranked system using approximation by finite ranked systems. This copy is called an *approximative version* of the infinite ranked system. We also develop comparison techniques for infinite systems, which parallel similar techniques for finite systems from Chapter 4. Finally, we show that if we take a infinite classical system and rank it, the resulting infinite ranked system will, in fact, be the approximative version. This allows us to use the results of this chapter not only for infinite ranked systems, but also for infinite classical systems.

### *7.3.1 Existence result and some properties*

First, we state a formal definition of an infinite ranked system of competing Brownian particles.

**Definition 32.** Fix parameters $g_1, g_2, \ldots \in \mathbb{R}$, $\sigma_1, \sigma_2, \ldots > 0$ and $(q_n^{\pm})_{n\ge 1}$ such that

$$q_{n+1}^{+} + q_n^{-} = 1, \quad 0 < q_n^{\pm} < 1, \quad n = 1, 2, \ldots$$

Consider an $\mathbb{R}^{\infty}$-valued process $Y = (Y(t), t \ge 0)$ with continuous adapted components and continuous adapted real-valued processes $L_{(k,k+1)} = (L_{(k,k+1)}(t), t \ge 0)$, $k = 1, 2, \ldots$ (for convenience, let $L_{(0,1)} \equiv 0$), with the following properties:

(i) $Y_1(t) \le Y_2(t) \le Y_3(t) \le \ldots$ for $t \ge 0$;

(ii) for i.i.d. standard $(\mathcal{F}_t)_{t\ge 0}$-Brownian motions $B_1, B_2, \ldots$, we have:

$$Y_k(t) = Y_k(0) + g_k t + \sigma_k B_k(t) + q_k^{+} L_{(k-1,k)}(t) - q_k^{-} L_{(k,k+1)}(t), \quad k = 1, 2, \ldots, \quad t \ge 0;$$

(iii) each process $L_{(k,k+1)}$ is nondecreasing, $L_{(k,k+1)}(0) = 0$ and

$$\int_0^{\infty} (Y_{k+1}(t) - Y_k(t))\, \mathrm{d}L_{(k,k+1)}(t) = 0, \quad k = 1, 2, \ldots$$

The last equation means that $L_{(k,k+1)}$ can increase only when $Y_k(t) = Y_{k+1}(t)$.

Then the process $Y$ is called an *infinite ranked system of competing Brownian particles* with *drift coefficients* $(g_k)_{k\ge1}$, *diffusion coefficients* $(\sigma_k^2)_{k\ge1}$, and *parameters of collisions* $(q_k^{\pm})_{k\ge1}$. The process $Y_k = (Y_k(t), t \ge 0)$ is called the *$k$th ranked particle*. The $\mathbb{R}_+^{\infty}$-valued process $Z = (Z(t), t \ge 0)$, $Z(t) = (Z_k(t))_{k\ge1}$, defined by

$$Z_k(t) = Y_{k+1}(t) - Y_k(t), \quad k = 1, 2, \dots, \quad t \ge 0,$$

is called the *gap process*. The process $L_{(k,k+1)}$ is called the *local time of collision* between $Y_k$ and $Y_{k+1}$. If $Y(0) = y$, then we say that this system $Y$ *starts from $y$*. The processes $B_1, B_2, \dots$ are called *driving Brownian motions*.

We can reformulate Proposition 7.2.4 (ii) as follows: take an infinite classical system $X = (X_i)_{i\ge1}$ of competing Brownian particles with drift coefficients $(g_n)_{n\ge1}$ and diffusion coefficients $(\sigma_n^2)_{n\ge1}$. Rank it: in other words, switch from named particles $X_i, i \ge 1$, to ranked particles $Y_k, k \ge 1$. The resulting system $Y = (Y_k)_{k\ge1}$ is an infinite ranked system of competing Brownian particles with drift coefficients $(g_n)_{n\ge1}$, diffusion coefficients $(\sigma_n^2)_{n\ge1}$, and parameters of collision

$$\overline{q}_n^{\pm} = 1/2, \quad n \ge 1.$$

In this chapter, we construct this infinite system by approximating it with finite systems of competing Brownian particles with the same parameters.

**Definition 33.** Using the notation from Definition 32, for every $N \ge 2$, let

$$Y^{(N)} = \left(Y_1^{(N)}, \dots, Y_N^{(N)}\right)'$$

be the system of $N$ ranked competing Brownian particles with drift coefficients $g_1, \dots, g_N$, diffusion coefficients $\sigma_1^2, \dots, \sigma_N^2$ and parameters of collision $(q_n^{\pm})_{1\le n\le N}$, driven by Brownian motions $B_1, \dots, B_N$. Suppose there exist limits

$$\lim_{N\to\infty} Y_k^{(N)}(t) =: Y_k(t),$$

which are uniform on every $[0,T]$, for every $k = 1, 2, \ldots$, and $Y = (Y_1, Y_2, \ldots)'$ turns out to be an infinite system of competing Brownian particles with parameters $(g_n)_{n\ge1}$, $(\sigma_n^2)_{n\ge1}$, $(q_n^{\pm})_{n\ge1}$. Then we say that $Y$ is an *approximative version* of this system.

The main result of this section is as follows.

**Theorem 7.3.1.** *Take a sequence of drift coefficients $(g_n)_{n\ge1}$, a sequence of diffusion coefficients $(\sigma_n^2)_{n\ge1}$, and a sequence of parameters of collision $(q_n^{\pm})_{n\ge1}$. Suppose that the initial conditions $y \in \mathbb{R}^\infty$ are such that $y_1 \le y_2 \le \ldots$, and*

$$\sum_{n=1}^{\infty} e^{-\alpha y_n^2} < \infty \quad \text{for all} \quad \alpha > 0.$$

*Assume that*

$$\inf_{n\ge1} g_n =: \overline{g} > -\infty, \quad \sup_{n\ge1} \sigma_n^2 =: \overline{\sigma}^2 < \infty, \tag{7.8}$$

*and there exists $n_0 \ge 1$ such that $q_n^+ \ge 1/2$ for $n \ge n_0$. Take any i.i.d. standard Brownian motions $B_1, B_2, \ldots$. Then there exists the approximative version of the infinite ranked system of competing Brownian particles with parameters $(g_n)_{n\ge1}, (\sigma_n^2)_{n\ge1}, (q_n^{\pm})_{n\ge1}$, starting from $y$, with driving Brownian motions $B_1, B_2, \ldots$*

*Remark* 22. We have not proved uniqueness for infinite ranked system from Theorem 7.3.1. We can so far only claim uniqueness for infinite *classical* systems. If we take the infinite ranked system from Theorem 7.3.1 with symmetric collisions ($q_n^{\pm} = 1/2,\ n = 1, 2, \ldots$), and impose the condition that this system must be the result of ranking a classical system, then we also get uniqueness. But without this special condition, we do not know that this is unique.

*Proof.* **Step 1.** $q_n^+ \ge 1/2$ *for all* $n \ge 1$. For $N \ge 2$, consider a ranked system

$$Y^{(N)} = \left(Y_1^{(N)}, \ldots, Y_N^{(N)}\right)',$$

of $N$ competing Brownian particles, with parameters $(g_n)_{1\le n\le N}$, $(\sigma_n^2)_{1\le n\le N}$, $(q_n^{\pm})_{1\le n\le N}$, starting from $\overline{Y}_k^{(N)}(0) = y_k,\ k = 1, \ldots, N$, with driving Brownian motions $B_1, B_2, \ldots, B_N$.

Define the new parameters of collision

$$\overline{q}_n^{\pm} = \frac{1}{2},\ n \ge 1.$$

Consider another ranked system

$$\overline{Y}^{(N)} = \left(\overline{Y}_1^{(N)}, \ldots, \overline{Y}_N^{(N)}\right)',$$

of $N$ competing Brownian particles, with parameters $(g_n)_{1\le n\le N}$, $(\sigma_n^2)_{1\le n\le N}$, $(\overline{q}_n^{\pm})_{1\le n\le N}$, starting from $\overline{Y}_k^{(N)}(0) = y_k,\ k = 1, \ldots, N$, with driving Brownian motions $B_1, B_2, \ldots, B_N$. By Corollary 4.3.8 and Remark 8, which correspond to [100, Corollary 3.9, Remark 7], we have:

$$\overline{Y}_k^{(N+1)}(t) \le \overline{Y}_k^{(N)}(t),\quad Y_k^{(N+1)}(t) \le Y_k^{(N)}(t),\quad k = 1, \ldots, N,\quad t \ge 0. \tag{7.9}$$

Since $q_n^+ \ge \overline{q}_n^+ = 1/2$ for $n = 1, \ldots, N$, by Corollary 4.3.11 from Chapter 4 (which corresponds to [100, Corollary 3.12]), we have:

$$\overline{Y}_k^{(N)}(t) \le Y_k^{(N)}(t),\quad t \ge 0,\quad k = 1, \ldots, N. \tag{7.10}$$

**Lemma 7.3.2.** *For every $T > 0$, we have a.s.*

$$\lim_{N\to\infty} \min_{0\le t\le T} \overline{Y}_1^{(N)}(t) = \inf_{N\ge 2} \min_{0\le t\le T} \overline{Y}_1^{(N)}(t) > -\infty.$$

The proof is postponed until the end of the proof of Theorem 7.3.1. Assuming we proved this lemma, let us continue the proof of Theorem 7.3.1.

**Step 2.** Assume we proved Lemma 7.3.2. For every $k \ge 1$, $t \ge 0$, $N \ge k$, we have:

$$Y_k^{(N)}(t) \ge \overline{Y}_k^{(N)}(t) \ge \overline{Y}_1^{(N)}(t) \ge \lim_{N\to\infty} \overline{Y}_1^{(N)}(t).$$

By (7.9), there exists a finite pointwise limit

$$Y_k(t) := \lim_{N\to\infty} Y_k^{(N)}(t). \tag{7.11}$$

Now, let $L^{(N)} = \left(L_{(1,2)}^{(N)}, \ldots, L_{(N-1,N)}^{(N)}\right)'$ be the vector of local times for the system $Y^{(N)}$.

**Lemma 7.3.3.** *There exist limits*

$$L_{(k,k+1)}(t) := \lim_{N\to\infty} L^{(N)}_{(k,k+1)}(t),$$

*for each* $k \ge 1$*, uniform on every* $[0,T]$*. The limit* $Y_k(t)$ *from* (7.11) *is also uniform on every* $[0,T]$ *for every* $k \ge 1$*.*

The proof of Lemma 7.3.3 is also postponed until the end of the proof of Theorem 7.3.1. Assuming we proved this lemma, let us complete the proof of Theorem 7.3.1 for the case when $q_n^+ \ge 1/2$ for all $n \ge 1$. Uniform limits of continuous functions are continuous; therefore, $L_{(k,k+1)}$ and $Y_k$ are continuous. We have:

$$Y_k^{(N)}(t) = y_k + g_k t + \sigma_k B_k(t) + q_k^+ L^{(N)}_{(k-1,k)}(t) - q_k^- L^{(N)}_{(k,k+1)}(t), \quad k = 1, 2, \dots, \quad t \ge 0.$$

Letting $N \to \infty$, we have:

$$Y_k(t) = y_k + g_k t + \sigma_k B_k(t) + q_k^+ L_{(k-1,k)}(t) - q_k^- L_{(k,k+1)}(t), \quad k = 1, 2, \dots, \quad t \ge 0.$$

Finally, let us show that $L_{(k,k+1)}$ and $Y_k$ satisfy the properties (i) - (iii) of Definition 32. Some of these properties follow directly from the uniform covergence and the corresponding properties for finite systems $Y^{(N)}$. The nontrivial part is to prove that $L_{(k,k+1)}$ can increase only when $Y_k = Y_{k+1}$. Suppose that for some $k \ge 1$ we have: $Y_k(t) < Y_{k+1}(t)$ for $t \in [\alpha, \beta] \subseteq \mathbb{R}_+$. By continuity, there exists $\varepsilon > 0$ such that $Y_{k+1}(t) - Y_k(t) \ge \varepsilon$ for $t \in [\alpha, \beta]$. By uniform convergence, for $N \ge N_0$ we have:

$$Y^{(N)}_{k+1}(t) - Y^{(N)}_k(t) \ge \frac{\varepsilon}{2}, \quad t \in [\alpha, \beta].$$

So $L^{(N)}_{(k,k+1)}$ is constant on $[\alpha, \beta]$: $L^{(N)}_{(k,k+1)}(\alpha) = L^{(N)}_{(k,k+1)}(\beta)$. This is true for all $N \ge N_0$. Letting $N \to \infty$, we get: $L_{(k,k+1)}(\alpha) = L_{(k,k+1)}(\beta)$. Therefore, $L_{(k,k+1)}$ is also constant on $[\alpha, \beta]$.

**Step 3.** Now, consider the case when $q_n^+ \ge 1/2$ only for $n \ge n_0$. It suffices to show that $(Y_k^{(N)}(t))_{N \ge k}$ is bounded from below (this is the crucial part of the proof). For $N \ge n_0 + 2$, consider the system

$$\tilde{Y}^{(N)} = \left(\tilde{Y}^{(N)}_{n_0+1}, \dots, \tilde{Y}^{(N)}_N\right)'$$

of $N - n_0$ competing Brownian particles with parameters $(g_n)_{n_0 < n \le N}$, $(\sigma_n^2)_{n_0 < n \le N}$, $(q_n^{\pm})_{n_0 < n \le N}$, starting from $(y_{n_0+1}, \ldots, y_N)'$, with driving Brownian motions $B_{n_0+1}, \ldots, B_N$. By Corollary 4.3.8 and Remark 8, which correspond to [100, Corollary 3.9, Remark 7], we have:

$$Y_k^{(N)}(t) \ge \tilde{Y}_k^{(N)}(t), \quad \text{for} \quad n_0 < k \le N \quad \text{and} \quad t \ge 0. \tag{7.12}$$

But for every $k > n_0$ and $t \in [0, T]$, the sequence $(\tilde{Y}_k^{(N)}(t))_{N > k}$ is bounded below: we proved this earlier in the proof of Theorem 7.3.1. Let us show that for every $t \in [0, T]$, the sequence $(\tilde{Y}_1^{(N)}(t))_{N \ge 2}$ is bounded below. Indeed, again applying Corollary 4.3.8 from Chapter 4, which corresponds to [100, Corollary 3.9], we get:

$$Z_k^{(n_0+1)}(t) \ge Z_k^{(N)}(t), \quad t \ge 0, \; k = 1, \ldots, n_0, \; N \ge n_0 + 2.$$

And

$$Y_1^{(N)}(t) = Y_{n_0+1}^{(N)}(t) - Z_{n_0}^{(N)}(t) - \ldots - Z_1^{(N)}(t) \ge Y_{n_0+1}^{(N)}(t) - Z_1^{(n_0+1)}(t) - \ldots - Z_{n_0}^{(n_0+1)}(t).$$

But $(Y_{n_0+1}^{(N)}(t))_{N \ge n_0+2}$ is bounded from below, and $Z_k^{(n_0+1)}(t)$ for $k = 1, \ldots, n_0$ are independent of $N$. Therefore, $(Y_1^{(N)}(t))_{N \ge 2}$ is bounded from below. The rest of the proof is the same as in the case when $q_n^+ \ge 1/2$ for all $n = 1, 2, \ldots$

*Proof of Lemma* 7.3.2. It suffices to show that, as $u \to \infty$, we have:

$$\mathbf{P}\left(\min_{0 \le t \le T} \overline{Y}_1^{(N)}(t) < -u\right) \to 0.$$

The ranked system $\overline{Y}^{(N)}$ has the same law as the result of ranking of a classical system

$$X^{(N)} = \left(X_1^{(N)}, \ldots, X_N^{(N)}\right)'$$

with the same parameters: drift coefficients $(g_n)_{1 \le n \le N}$, diffusion coefficients $(\sigma_n^2)_{1 \le n \le N}$, starting from $X^{(N)}(0) = (y_1, \ldots, y_N)'$. These components satisfy the following system of SDE:

$$\mathrm{d}X_i^{(N)}(t) = \sum_{k=1}^{N} 1(X_i^{(N)} \text{ has rank } k \text{ at time } t)\left(g_k \mathrm{d}t + \sigma_k \mathrm{d}W_i(t)\right), \quad i = 1, \ldots, N, \tag{7.13}$$

for some i.i.d. standard Brownian motions $W_1, \ldots, W_N$. In particular,

$$Y_1^{(N)}(t) \equiv \min_{i=1,\ldots,N} X_i^{(N)}(t).$$

Therefore,

$$\min_{0\le t\le T} Y_1^{(N)}(t) = \min_{1\le i\le N} \min_{0\le t\le T} X_i^{(N)}(t). \tag{7.14}$$

We can rewrite (7.13) as

$$X_i^{(N)}(t) = y_i + \int_0^t \beta_{N,i}(s)\mathrm{d}s + \int_0^t \rho_{N,i}(s)\mathrm{d}s,$$

where

$$\beta_{N,i}(t) := \sum_{k=1}^N 1(X_i^{(N)} \text{ has rank } k \text{ at time } t) g_k,$$

$$\rho_{N,i}(t) := \sum_{k=1}^N 1(X_i^{(N)} \text{ has rank } k \text{ at time } t) \sigma_k.$$

Because of (7.8), we get:

$$\beta_{N,i}(t) \ge \overline{g}, \quad |\rho_{N,i}(t)| \le \overline{\sigma}, \quad t \ge 0.$$

Therefore,

$$X_i^{(N)}(t) \ge y_i + \overline{g}T + \mathcal{M}_{N,i}(t), \quad \text{where } \ \mathcal{M}_{N,i}(t) := \int_0^t \rho_{N,i}(s)\mathrm{d}W_i(s)$$

is a continuous square-integrable martingale with quadratic variation

$$\langle \mathcal{M}_{N,i}\rangle_t = \int_0^t \rho_{N,i}^2(s)\mathrm{d}s \le \overline{s}^2 T, \quad t \in [0, T].$$

Let us make a time-change: for some standard Brownian motion $\overline{B}_i = (\overline{B}_i(s), s \ge 0)$, we have:

$$\mathcal{M}_{N,i}(t) = \overline{B}_i\left(\langle \mathcal{M}_{N,i}\rangle_t\right).$$

So

$$\min_{0\le t\le T} X_i^{(N)}(t) \ge y_i + (\overline{g}T)_- + \min_{[0,\overline{\sigma}^2 T]} \overline{B}_i(s).$$

Therefore,

$$\begin{aligned}
&\mathbf{P}\left(\min_{0\le t\le T} X_i^{(N)}(t) < -u\right) \le \mathbf{P}\left(\min_{0\le t\le \overline{\sigma}^2 T} \overline{B}_i(t) < -u - y_i - (\overline{g}T)_-\right) \\
&= \mathbf{P}\left(\max_{0\le t\le \overline{\sigma}^2 T} \overline{B}_i(t) > u + y_i + (\overline{g}T)_-\right) = 2\mathbf{P}\left(\overline{B}_i(\overline{\sigma}^2 T) > u + y_i + (\overline{g}T)_-\right) \\
&= 2\Psi\left(\frac{u + y_i + (\overline{g}T)_-}{\overline{\sigma}\sqrt{T}}\right).
\end{aligned}$$

From (7.14), we have:

$$\mathbf{P}\left(\min_{0\le t\le T} \overline{Y}_1^{(N)}(t) < -u\right) \le \sum_{i=1}^N \mathbf{P}\left(\min_{0\le t\le T} X_i^{(N)}(t) < -u\right) \le 2\sum_{i=1}^N \Psi\left(\frac{u + y_i + (\overline{g}T)_-}{\overline{\sigma}\sqrt{T}}\right).$$

Since the sequence of real numbers $(\overline{Y}_1^{(N)}(t))_{N\ge 2}$ is nonincreasing for every $t \ge 0$, using Lemma 7.7.1, we have:

$$\begin{aligned}
&\mathbf{P}\left(\min_{0\le t\le T} \overline{Y}_1^{(N)}(t) < -u\right) = \lim_{N\to\infty} \mathbf{P}\left(\min_{0\le t\le T} \overline{Y}_1^{(N)}(t) < -u\right) \\
&\le \lim_{N\to\infty} 2\sum_{i=1}^N \Psi\left(\frac{u + y_i + (\overline{g}T)_-}{\overline{\sigma}\sqrt{T}}\right) = 2\sum_{i=1}^\infty \Psi\left(\frac{u + y_i + (\overline{g}T)_-}{\overline{\sigma}\sqrt{T}}\right) < \infty.
\end{aligned}$$

Let $u \to \infty$. Then

$$\frac{y_i + (\overline{g}T)_- + u}{\overline{\sigma}\sqrt{T}} \to \infty, \quad \Psi\left(\frac{y_i + (\overline{g}T)_- + u}{\overline{\sigma}\sqrt{T}}\right) \to 0.$$

Applying Lebesgue dominated convergence theorem to this series (and using the fact that $\Psi$ is decreasing), we get:

$$\sum_{i=1}^\infty \Psi\left(\frac{u + y_i + (\overline{g}T)_-}{\overline{\sigma}\sqrt{T}}\right) \to 0 \quad \text{as} \quad u \to \infty.$$

This completes the proof of Lemma 7.3.2.

*Proof of Lemma* 7.3.3. Applying Corollary 4.3.8 (which corresponds to [100, Corollary 3.9]) again, we have:

$$L_{(k,k+1)}^{(N)}(t) - L_{(k,k+1)}^{(N)}(s) \le L_{(k,k+1)}^{(M)}(s) - L_{(k,k+1)}^{(M)}(s), \quad 0 \le s \le t,\ 1 \le k < N < M; \tag{7.15}$$

Note that $y_k = Y_k^{(N)}(0),\ N \ge k$, does not depend on $N$, by construction of the system. So

$$Y_1^{(N)}(t) = y_1 + g_1 t + \sigma_1 B_1(t) - q_1^- L_{(1,2)}^{(N)}(t).$$

Since $Y_1^{(N)}(t) \to Y_1(t)$ and $q_1^- > 0$: the sequence $(L_{(1,2)}^{(N)}(t))_{N\ge 2}$ has a limit $L_{(1,2)}(t) := \lim_{N\to\infty} L_{(1,2)}^{(N)}(t)$ for every $t \ge 0$. Letting $M \to \infty$ in (7.15), we get: $L_{(1,2)}(t) - L_{(1,2)}(s) \ge L_{(1,2)}^{(N)}(t) - L_{(1,2)}^{(N)}(s)$, for $0 \le s \le t$. Rewrite this as

$$L_{(1,2)}(t) - L_{(1,2)}^{(N)}(t) \ge L_{(1,2)}(s) - L_{(1,2)}^{(N)}(s). \tag{7.16}$$

But we also have: $(L_{(1,2)}^{(N)}(t))_{N\ge 2}$ is nondecreasing. Therefore,

$$L_{(1,2)}(s) - L_{(1,2)}^{(N)}(s) \ge 0. \tag{7.17}$$

In addition,

$$L_{(1,2)}^{(N)}(t) \to L_{(1,2)}(t) \ \text{ as } \ N \to \infty. \tag{7.18}$$

Combining (7.16), (7.17), (7.18), we get: $L_{(1,2)}^{(N)}(s) \to L_{(1,2)}(s)$, as $N \to \infty$, uniformly on every $[0,t]$. Therefore, letting $N \to \infty$ in (7.9), we get:

$$Y_1(t) = y_1 + g_1 t + \sigma_1 B_1(t) - q_1^- L_{(1,2)}(t), \quad t \ge 0,$$

and $Y_1^{(N)}(s) \to Y_1(s)$ uniformly on every $[0,t]$. Since $Y_1^{(N)}$ and $L_{(1,2)}^{(N)}$ are continuous for every $N \ge 2$, and the uniform limit of continuous functions is continuous, we conclude that the functions $Y_1$ and $L_{(1,2)}$ are also continuous. Now,

$$Y_2^{(N)}(t) = y_2 + g_2 t + \sigma_2 B_2(t) + q_2^+ L_{(1,2)}^{(N)}(t) - q_2^- L_{(2,3)}^{(N)}(t), \quad t \ge 0.$$

But

$$Y_2^{(N)}(t) \to Y_2(t) \ \text{ and } \ L_{(1,2)}^{(N)}(t) \to L_{(1,2)}(t) \ \text{ as } \ N \to \infty.$$

Since $q_2^- > 0$, we have: there exists a limit $L_{(2,3)}(t) := \lim\limits_{N\to\infty} L_{(2,3)}^{(N)}(t)$. Similarly to $L_{(1,2)}^{(N)} \to L_{(1,2)}$, we prove that this convergence is uniform on every $[0,t]$. So $Y_2^{(N)} \to Y_2$ as $N \to \infty$ uniformly on every $[0,t]$. Thus $Y_2$ and $L_{(2,3)}$ are continuous.

Analogously, we can prove that for every $k \ge 1$, the limits

$$L_{(k,k+1)}(t) = \lim_{N\to\infty} L^{(N)}_{(k,k+1)}(t) \ \text{ and } \ Y_k(t) = \lim_{N\to\infty} Y_k^{(N)}(t)$$

exist and are uniform on every $[0,T]$. This completes the proof of Lemma 7.3.3, and with it the proof of Theorem 7.3.1. □

Let us now prove some additional properties of this newly constructed approximative version of an infinite system of competing Brownian particles. These are analogous to the properties of an infinite classical system of competing Brownian particles, stated in Proposition 7.2.4 above.

**Lemma 7.3.4.** *For an approximative version of an infinite ranked system from Theorem* 7.3.1*, we have:*

*(i) for every $y \in \mathbb{R}$ and $T > 0$, a.s. there are only finitely many $k \ge 1$ such that*

$$\min_{0\le t\le T} Y_k(t) \le y;$$

*(ii) for every $t \ge 0$, we have:*

$$\sum_{k=1}^{\infty} e^{-\alpha Y_k^2(t)} < \infty \ \textit{ for every } \ \alpha > 0.$$

*Proof.* **Step 1.** First, consider the case $q_n^+ \ge 1/2$ for all $n \ge 1$.

(i) It suffices to show that

$$\sum_{n=1}^{\infty} \mathbf{P}\left(\min_{0\le t\le T} Y_k(t) < u\right) < \infty,$$

and then apply the Borel-Cantelli lemma. But for every $k \ge 1$, we have:

$$Y_k(t) = \lim_{N\to\infty} Y_k^{(N)}(t), \tag{7.19}$$

uniformly on $[0,T]$. Therefore,

$$\mathbf{P}\left(\min_{0\le t\le T} Y_k(t) < u\right) = \lim_{N\to\infty} \mathbf{P}\left(\min_{0\le t\le T} Y_k^{(N)}(t) < u\right). \tag{7.20}$$

We claim that the following estimate is true:

$$\mathbf{P}\left(\min_{0\le t\le T} Y_k^{(N)}(t) < u\right) \le 2\sum_{j=k}^{N}\Psi\left(\frac{(\overline{g}T)_- + y_j - u}{\overline{\sigma}\sqrt{T}}\right). \tag{7.21}$$

Assuming that we proved (7.21), let us complete the proof. Letting $N \to \infty$ and using (7.20), we get:

$$\mathbf{P}\left(\min_{0\le t\le T} Y_k(t) < u\right) \le 2\sum_{j=k}^{\infty}\Psi\left(\frac{(\overline{g}T)_- + y_j - u}{\overline{\sigma}\sqrt{T}}\right).$$

Now, let $\mathcal{K} := \max\{k \mid \min_{[0,T]} Y_k < u\}$. Then

$$\mathbf{P}(\mathcal{K} \ge k) = \mathbf{P}\left(\min_{0\le t\le T} Y_k(t) < u\right) \le 2\sum_{j=k}^{\infty}\Psi\left(\frac{(\overline{g}T)_- + y_j - u}{\overline{\sigma}\sqrt{T}}\right) \to 0$$

as $k \to \infty$. Therefore, $\mathcal{K} < \infty$ a.s., which is equivalent to (i).

Now, let us show (7.21). It suffices to show that

$$\mathbf{P}\left(\min_{0\le t\le T} \overline{Y}_k^{(N)}(t) < u\right) \le 2\sum_{j=k}^{N}\Psi\left(\frac{(\overline{g}T)_- + y_j - u}{\overline{\sigma}\sqrt{T}}\right),$$

because of (7.10). Now, consider another ranked system

$$Y' = (Y_k', \ldots, Y_N')'$$

of $N-k+1$ competing Brownian particles, with drift coefficients $(g_n)_{k\le n\le N}$, diffusion coefficients $(\sigma_n^2)_{k\le n\le N}$, and symmetric collisions, so the parameters of collisions are $\overline{q}_n^{\pm} = 1/2,\ k \le n \le N$, with driving Brownian motions $B_k, \ldots, B_N$ (where $B_1, \ldots, B_N$ are driving Brownian motions for $Y^{(N)}$), starting from $(y_k, \ldots, y_N)'$. Then by Remark 8 from Chapter 4 we have:

$$Y_j'(t) \le \overline{Y}_j(t),\quad j = k, \ldots, N,\quad t \ge 0.$$

So it suffices to show that

$$\mathbf{P}\left(\min_{0\le t\le T} Y_k'(t) < u\right) \le 2\sum_{j=k}^{N}\Psi\left(\frac{(\overline{g}T)_- + y_j - u}{\overline{\sigma}\sqrt{T}}\right),$$

but this is done in the same way as in the proof of Theorem 7.2.1. This completes the proof of (7.21).

(ii) For every $k = 1, 2, \ldots$ and every $\alpha > 0$, we have:

$$\sum_{j=1}^{k} e^{-\alpha Y_j^2(t)} = \lim_{N\to\infty} \sum_{j=1}^{k} e^{-\alpha\left[Y_j^{(N)}\right]^2} \le \lim_{N\to\infty} \sum_{j=1}^{k} e^{-\alpha\left[\overline{Y}_j^{(N)}\right]^2},$$

where we use the notation from the proof of Theorem 7.3.1. The latter limit for different $N$ forms a sequence of real numbers which is nondecreasing, because $(\overline{Y}_j^{(N)}(t))_{N\ge j}$ is nonincreasing. In the proof of Lemma 7.3.2, we showed that for any $T > 0$, and $i = 1, 2, \ldots$

$$Y_i^{(N)}(T) \ge \min_{0\le t\le T} Y_i^{(N)}(t) \ge y_i + (\overline{g}T)_- + \min_{0\le t\le \overline{\sigma}^2 T} \overline{B}_i(t),$$

where $\overline{B}_1, \overline{B}_2, \ldots$ are standard Brownian motions. In the proof of Lemma 7.2.4 (ii), see (7.7), we showed that for all but finitely many $i \ge 1$,

$$y_i + (\overline{g}T)_- + \min_{0\le t\le \overline{\sigma}^2 T} \overline{B}_i(t) \ge \frac{1}{2} y_i + (\overline{g}T)_-.$$

By Lemma 7.7.1, for all $\alpha > 0$,

$$\sum_{i=i_0}^{\infty} e^{-\alpha((1/2)y_i + (\overline{g}T)_-)^2} = C < \infty,$$

and this constant $C$ depends only on $T, \alpha, i_0$ and $(y_j)_{j\ge1}$. (It is, however, random, because $i_0$ is random.) Therefore, for all $k \ge i_0$,

$$\sum_{j=i_0}^{k} e^{-\alpha Y_j^2(t)} \le C.$$

Let $k \to \infty$ and complete the proof of (ii).

**Step 2.** Now, let us prove (i) and (ii) for the general case. We still have (7.19). Recall the definition of a finite ranked system

$$\tilde{Y}^{(N)} = \left(\tilde{Y}_{n_0+1}^{(N)}, \ldots, \tilde{Y}_N^{(N)}\right)'$$

of competing Brownian particles from the proof of Theorem 7.3.1, Step 3. In Theorem 7.3.1, we prove that for every $k > n_0$, as $N \to \infty$, uniformly on $[0, T]$ we have:

$$\tilde{Y}_k^{(N)}(t) \to \tilde{Y}_k(t),$$

where $\tilde{Y} = (\tilde{Y}_k)_{k>n_0}$ is an infinite ranked system of competing Brownian particles with parameters $(g_n)_{n>n_0}, (\sigma_n^2)_{n>n_0}, (q_n^{\pm})_{n>n_0}$. But $q_n^+ \ge 1/2$ for all $n > n_0$, and therefore the system $\tilde{Y}$ satisfies the statements (i) and (ii) of Lemma 7.3.4. Letting $N \to \infty$ in (7.12), we get:

$$Y_k(t) \ge \tilde{Y}_k(t), \quad t \in [0, T], \quad n_0 < k \le N.$$

Therefore, the system $(Y_k)_{k\ge1}$ also satisfies the statements (i) and (ii) of Lemma 7.3.4. $\square$

Let us also state another useful lemma; the proof is postponed until Section 7.6.

**Lemma 7.3.5.** *Consider the infinite system from Theorem* 7.3.1*. Then for every $t > 0$ a.s. the vector $Y(t) = (Y_k(t))_{k\ge1}$ has no ties.*

We developed comparison techniques for finite systems of competing Brownian particles in Chapter 4. These techniques also work for infinite ranked systems, provided we take their approximative versions. By taking limits as the number $n$ of particles goes to infinity, we can formulate the same comparison results for these two infinite systems. Let us give an example. The proof trivially follows from Corollary 4.3.10 from Chapter 4 (which corresponds to [100, Corollary 3.11]) and is therefore omitted.

**Corollary 7.3.6.** *Take two approximative versions $Y$ and $\overline{Y}$ of an infinite system of competing Brownian particles with parameters*

$$(g_n)_{n\ge1}, \ (\sigma_n^2)_{n\ge1}, \ (q_n^{\pm})_{n\ge1},$$

*with the same driving Brownian motions. Let $Z$ and $\overline{Z}$ be the corresponding gap processes. Then:*

*(i) If $Y(0) \le \overline{Y}(0)$, then $Y(t) \le \overline{Y}(t)$, $t \ge 0$.*

*(ii) If $Z(0) \le \overline{Z}(0)$, then $Z(t) \le \overline{Z}(t)$, $t \ge 0$.*

The next corollary is a counterpart of Corollary 4.3.12 from Chapter 4 (which is also mentioned as [100, Corollary 3.13]).

**Corollary 7.3.7.** *Take two approximative versions $Y$ and $\overline{Y}$ of an infinite system of competing Brownian particles with parameters*

$$(g_n)_{n\ge1},\ (\sigma_n^2)_{n\ge1},\ (q_n^{\pm})_{n\ge1},$$

*and*

$$(\overline{g}_n)_{n\ge1},\ (\sigma_n^2)_{n\ge1},\ (q_n^{\pm})_{n\ge1},$$

*with the same driving Brownian motions, starting from the same initial conditions. Let $Z$ and $\overline{Z}$ be the corresponding gap processes. Then:*

*(i) If $g_n \le \overline{g}_n,\ n = 1, 2, \ldots$, then $Y(t) \le \overline{Y}(t)$, $t \ge 0$;*

*(ii) If $g_{n+1} - g_n \le \overline{g}_{n+1} - \overline{g}_n,\ n = 1, 2, \ldots$, then $Z(t) \le \overline{Z}(t)$, $t \ge 0$.*

*Remark* 23. If, in each of these two corollaries, we remove the requirement that the two systems have the same driving Brownian motions, then we have stochastic ordering instead of pathwise ordering. The same applies to Corollary 7.3.6 if we change a.s. comparison $Y(0) \le \overline{Y}(0)$ ($Z(0) \le \overline{Z}(0)$, respectively) to stochastic comparison of these initial conditions.

### *7.3.2 Approximative version of an infinite classical system*

Now, consider infinite classical systems of competing Brownian particles. If you rank the named particles in it, then, as shown in Proposition 7.2.4, we get an infinite ranked system $Y = (Y_n)_{n\ge1}$ of competing Brownian particles in the sense of Definition 32, with $q_n^{\pm} = 1/2$, $n = 1, 2, \ldots$ We learned how to construct infinite ranked systems from Theorem 7.3.1: by approximating them with finite systems. We know from [105] and [59] that infinite classical systems $X = (X_n)_{n\ge1}$ can also be constructed, for the case when the sequences $(g_n)_{n\ge1}$ and $(\sigma_n^2)_{n\ge1}$ stabilize starting from some $n_0$. In this case, these classical systems exist and are unique in the weak sense.

In this section, we prove weak existence (but not uniqueness) of an infinite classical system in a more general case: when the sequences $(g_n)_{n\ge1}$ and $(\sigma_n^2)_{n\ge1}$ are bounded. We construct an approximative version of such system, just like we did above for an infinite ranked system. We also show that if we rank the approximative version of an infinite classical system, we get the approximative version of an infinite ranked system. In particular, the infinite systems constructed in Proposition 7.2.1 (for which weak existence and uniqueness hold) from [105] and [59] turn out to be approximative versions. In particular, the infinite Atlas model is an approximative version.

In the next section, we get some results for approximative versions of infinite ranked systems. This connection allows us to apply these results to infinite classical systems; in particular, the ones constructed in Proposition 7.2.1. This is how we get Theorem 1.4.1 from Chapter 1.

We were able to prove that the answer is affirmative, if $(g_n)_{n\ge1}$ is bounded. This allows us to apply later results about stationary distributions and convergence (proved using the same approximation techniques), to classical systems $X$ as well as ranked systems $Y$, provided that the collisions are symmetric: $q_n^{\pm} = 1/2,\ n = 1, 2, \ldots$

To put this another way: It is known that infinite classical systems exist and are unique in the weak sense. If you rank named particles in this system, you get an infinite ranked system. At the same time, we have constructed approximative versions of infinite ranked systems, and we can use comparison techniques for them. The following theorem tells that the infinite ranked system emerging from the classical innfinite system is in fact the approximative version. So we can apply the whole range of comparison techniques and the results of this chapter to infinite classical systems. The proof can be found in Section 7.6.

**Theorem 7.3.8.** *Fix parameters $(g_n)_{n\ge1}$ and $(\sigma_n^2)_{n\ge1}$ such that*

$$\sup_{n\ge1} |g_n| < \infty \quad \textit{and} \quad \sup_{n\ge1} \sigma_n^2 < \infty.$$

*Fix a rankable initial condition $x \in \mathbb{R}^\infty$, such that*

$$\sum_{n=1}^{\infty} e^{-\alpha x_n^2} < \infty \quad \text{for all} \quad \alpha > 0.$$

*Consider a copy $X^{(N)} = (X^{(N)}(t), t \ge 0)$ of a classical system of $N$ competing Brownian particles with drift coefficients $(g_n)_{1\le n\le N}$ and diffusion coefficients $(\sigma_n^2)_{1\le n\le N}$, which starts from $X^{(N)}(0) = [x]_N$. Let $Y^{(N)}$ be the corresponding ranked system. Then*

$$X_k^{(N)} \Rightarrow X_k \quad as \quad N \to \infty, \tag{7.22}$$

*for every $k \ge 1$, in the topology of $C[0,T]$ for every $T > 0$, where $X = (X_k)_{k\ge1}$ turns out to be a infinite classical system of competing Brownian particles with parameters $(g_n)_{n\ge1}$ and $(\sigma_n^2)_{n\ge1}$. Moreover, let $Y^{(N)}$ be the ranked system $X^{(N)}$. Then*

$$Y_k^{(N)} \Rightarrow Y_k, \quad as \quad N \to \infty, \tag{7.23}$$

*for every $k \ge 1$, in the topology of $C[0,T]$ for every $T > 0$, where $(Y_k)_{k\ge1}$ is the ranked system $(X_k)_{k\ge1}$.*

## 7.4 The Gap Process: Stationary Distributions and Weak Convergence

In this section, we prove Theorem 1.4.1 and similar results for general infinite systems of competing Brownian particles. First, we construct a stationary distribution $\pi$ for the gap process $Z = (Z(t), t \ge 0)$ of such system. Then we prove that: (i) any weak limit point of the gap process $Z(t)$ as $t \to \infty$ is stochastically dominated by $\pi$, and (ii) if the initial gaps $Z(0)$ are stochastically larger than $\pi$, then $Z(t) \Rightarrow \pi$ as $t \to \infty$ (if only we consider an approximative version of the system).

### *7.4.1 Stationary Distributions.*

Consider again an infinite system $Y$ of competing Brownian particles with parameters $(g_n)_{n\ge1}$, $(\sigma_n^2)_{n\ge1}$, $(q_n^\pm)_{n\ge1}$. Let $Z$ be its gap process.

**Definition 34.** Let $\pi$ be a probability measure on $\mathbb{R}_+^\infty$. We say that $\pi$ *is a stationary distribution* for the gap process for the system above if there exists a version $Y$ of this system such that for every $t \ge 0$, we have: $Z(t) \backsim \pi$.

Let us emphasize that in this chapter, we do not study uniqueness and Markov property. We simply construct a copy of the system with required properties.

We already know from [89] that

$$\pi_\infty = \bigotimes_{n=1}^{\infty} \mathcal{E}(2)$$

is a stationary distribution for the gap process of the infinite Atlas model:

$$g_1 = 1, g_2 = g_3 = \ldots = 0,\ \sigma_1 = \sigma_2 = \ldots = 1,\ q_1^{\pm} = q_2^{\pm} = \ldots = \frac{1}{2}.$$

Here, we find stationary distributions for other infinite systems of competing Brownian particles and prove convergence results for them. In addition, we show how to prove the main result of [89] in an arguably more natural way.

Consider, for each $N \ge 2$, the ranked system of $N$ competing Brownian particles with parameters $(g_n)_{1\le n\le N}, (\sigma_n^2)_{1\le n\le N}, (q_n^{\pm})_{1\le n\le N}$. Assume that these parameters are such that for $N > N_0$ the gap process has a stationary distribution. According to Proposition 3.5.1, this is the case when

$$[R^{(N)}]^{-1}\mu^{(N)} < 0,$$

where

$$R^{(N)} = \begin{bmatrix} 1 & -q_2^- & 0 & \ldots & 0 \\ -q_2^+ & 1 & -q_3^- & \ldots & 0 \\ 0 & -q_3^+ & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

is an $(N-1)\times(N-1)$ matrix, and

$$\mu^{(N)} = (g_2 - g_1, g_3 - g_2, \ldots, g_N - g_{N-1})'.$$

Let $B_1, B_2, \ldots$ be i.i.d. standard Brownian motions. Let $\pi^{(N)}$ be the stationary distribution on $\mathbb{R}_+^{N-1}$. Let $z^{(N)} \backsim \pi^{(N)}$ be an $\mathcal{F}_0$-measurable random variable and consider the system $\overline{Y}^{(N)}$ of $N$ ranked competing Brownian particles with parameters $(g_n)_{1\le n\le N}, (\sigma_n^2)_{1\le n\le N}, (q_n^{\pm})_{1\le n\le N}$, starting from $(0, z_1^{(N)}, \ldots, z_1^{(N)} + \ldots + z_{N-1}^{(N)})'$, driven by $B_1, \ldots, B_N$.

**Lemma 7.4.1.** $[\pi^{(N+1)}]_{N-1} \preceq \pi^{(N)}$.

*Proof.* Take a system $\tilde{Y}^{(N)}$ of $N$ competing Brownian particles with the same parameters and the same driving Brownian motions as $\overline{Y}^{(N)}$, but starting from $(0, \ldots, 0)' \in \mathbb{R}^N$. Take another system $\tilde{Y}^{(N+1)}$ of $N+1$ competing Brownian particles with parameters

$$(g_n)_{1\le n\le N+1}, \quad (\sigma_n^2)_{1\le n\le N+1}, \quad (q_n^{\pm})_{1\le n\le N+1},$$

and driving Brownian motions $B_1, \ldots, B_{N+1}$, starting from $(0, 0, \ldots, 0)' \in \mathbb{R}^{N+1}$. Then by Corollary 4.3.8 from Chapter 4, which corresponds to [100, Corollary 3.9], the corresponding gap processes $\tilde{Z}^{(N)}$ and $\tilde{Z}^{(N+1)}$ satisfy

$$\tilde{Z}^{(N)}(t) \ge [\tilde{Z}^{(N+1)}(t)]_{N-1}, \quad t \ge 0.$$

But

$$\tilde{Z}^{(N)}(t) \Rightarrow \pi^{(N)}, \ \tilde{Z}^{(N+1)}(t) \Rightarrow \pi^{(N+1)}, \quad t \to \infty.$$

So $[\pi^{(N+1)}]_{N-1} \preceq \pi^{(N)}$. $\square$

Without loss of generality, by changing the probability space we can take $z^{(N)} \backsim \pi^{(N)}$ such that a.s. $[z^{(N+1)}]_{N-1} \le z^{(N)}$, for $N > N_0$. In other words, $z_k^{(N+1)} \le z_k^{(N)}$, $k = 1, \ldots, N-1$. Since all $z_k^{(N)}$ are always nonnegative, there exists

$$z_k = \lim_{N\to\infty} z_k^{(N)}, \quad k \ge 1.$$

Denote by $\pi$ the distribution of $(z_1, z_2, \ldots)$ on $\mathbb{R}_+^{\infty}$. Then $\pi$ becomes a prospective stationary distribution for the gap process for the infinite system of competing Brownian particles.

Equivalently, we can define $\pi$ as follows: for every $N \ge 1$, let

$$[\pi^{(M)}]_{N-1} \ \Rightarrow \ \rho^{(N)}, \quad M \to \infty.$$

These finite-dimensional distributions $\rho^{(N)}$ are *consistent*:

$$[\rho^{(N+1)}]_{N-1} = \rho^{(N)}, \quad N \ge 1.$$

So by Kolmogorov's theorem there exists a unique distribution $\pi$ on $\mathbb{R}_+^\infty$ such that $[\pi]_{N-1} = \rho^{(N)}$ for all $N \ge 1$.

The next lemma allows us to rewrite the condition (7.1) in terms of the gap process. The proof is postponed until Section 7.5 (Appendix).

**Lemma 7.4.2.** *For a sequence $y = (y_n)_{n\ge1} \in \mathbb{R}^\infty$ such that $y_n \le y_{n+1},\ n \ge 1$, let $z = (z_n)_{n\ge1} \in \mathbb{R}^\infty$ be defined by $z_n = y_{n+1} - y_n,\ n \ge 1$. Then $y$ satisfies* (7.1) *if and only if $z$ satisfies*

$$\sum_{n=1}^{\infty} \exp\left(-\alpha(z_1 + \ldots + z_n)^2\right) < \infty \quad \text{for all} \quad \alpha > 0. \tag{7.24}$$

Now, let us state one of the two main results of this section.

**Theorem 7.4.3.** *Consider an infinite system of competing Brownian particles with parameters*

$$(g_n)_{n\ge1},\ (\sigma_n^2)_{n\ge1},\ (q_n^{\pm})_{n\ge1}.$$

*Assume*

$$[R^{(N)}]^{-1}\mu^{(N)} < 0, \quad N > N_0.$$

*Suppose that*

$$\inf_{n\ge1} g_n > -\infty, \quad \sup_{n\ge1} \sigma_n^2 < \infty,$$

*and for some $n_0 \ge 1$ we have:*

$$q_n^+ \ge \frac{1}{2}, \quad n \ge n_0.$$

*Assume that for $N > N_0$ we have: $[R^{(N)}]^{-1}\mu^{(N)} < 0$, so that we construct the distribution $\pi$. Assume, in addition, that $\pi$-a.s.* (7.24)*. Then we can construct an approximative version of the infinite system of competing Brownian particles with parameters*

$$(g_n)_{n\ge1},\ (\sigma_n^2)_{n\ge1},\ (q_n^{\pm})_{n\ge1},$$

*such that $\pi$ is a stationary distribution for the gap process.*

*Remark* 24. For finite systems of ranked competing Brownian particles, if a stationary distribution for the gap process exists, it is unique. This was already mentioned in Section 3.5. For infinite system, we do not know whether this is true.

*Proof.* **Step 1.** Using the notation of Theorem 7.3.1, we have:

$$Y_k^{(N)} \to Y_k, \quad N \to \infty,$$

for every $k \ge 1$, uniformly on every $[0, T]$. Now, let

$$\overline{Y}^{(N)} = \left(\overline{Y}_1^{(N)}, \ldots, \overline{Y}_N^{(N)}\right)'$$

be the ranked system of $N$ competing Brownian particles, which has the same parameters and driving Brownian motions as $Y^{(N)} = \left(Y_1^{(N)}, \ldots, Y_N^{(N)}\right)'$, but starts from $(0, z_1^{(N)}, z_1^{(N)} + z_2^{(N)}, \ldots, z_1^{(N)} + z_2^{(N)} + \ldots + z_{N-1}^{(N)})'$, rather than $(0, z_1, z_1 + z_2, \ldots, z_1 + z_2 + \ldots + z_{N-1})'$. In other words, the gap process $\overline{Z}^{(N)}$ of the system $\overline{Y}^{(N)}$ is in its stationary regime: $\overline{Z}^{(N)}(t) \backsim \pi^{(N)}$, $t \ge 0$. It suffices to show that a.s., as $N \to \infty$, for all $t \ge 0$ and $k \ge 1$, we have:

$$Y_k(t) = \lim_{N\to\infty} Y_k^{(N)}(t). \tag{7.25}$$

Indeed, assuming that we have already shown this, the proof can be quickly finished, as follows: for every $t \ge 0$ and $k = 1, 2, \ldots$, a.s.

$$\overline{Z}_k^{(N)}(t) = \overline{Y}_{k+1}^{(N)}(t) - \overline{Y}_k^{(N)}(t) \to Z_k(t) = Y_{k+1}(t) - Y_k(t), \quad N \to \infty.$$

Therefore, for every $t \ge 0$ and $N \ge 2$, a.s. we have:

$$\left(\overline{Z}_1^{(M)}(t), \ldots, \overline{Z}_{N-1}^{(M)}(t)\right)' \to \left(Z_1(t), \ldots, Z_{N-1}(t)\right)', \quad M \to \infty.$$

But

$$\overline{Z}^{(M)}(t) = \left(\overline{Z}_1^{(M)}(t), \ldots, \overline{Z}_{M-1}^{(M)}(t)\right)' \backsim \pi^{(M)}$$

for $M \ge 2$ and $t \ge 0$, and

$$[\pi^{(M)}]_{N-1} \ \Rightarrow \ [\pi]_{N-1}.$$

So for $N \ge 2$, $t \ge 0$ we have:

$$(Z_1(t), \ldots, Z_{N-1}(t))' \backsim [\pi]_{N-1}.$$

Thus, for $Z(t) := (Z_1(t), Z_2(t), \ldots)$, we have:

$$Z(t) \backsim \pi, \quad t \ge 0.$$

**Step 2.** Let us prove (7.25). First, since $z_1 \le z_1^{(N)}, \ldots, z_{N-1} \le z_{N-1}^{(N)}$, we have:

$$\begin{aligned} Y^{(N)}(0) =& (0, z_1, z_1 + z_2, \ldots, z_1 + z_2 + \ldots + z_{N-1})' \\ & \le \overline{Y}^{(N)}(0) = (0, z_1^{(N)}, z_1^{(N)} + z_2^{(N)}, \ldots, z_1^{(N)} + z_2^{(N)} + \ldots + z_{N-1}^{(N)})'. \end{aligned}$$

By Corollary 4.3.10(i) from Chapter 4, which corresponds to [100, Corollary 3.11(i)],

$$Y_k^{(N)}(t) \le \overline{Y}_k^{(N)}(t),\ t \ge 0,\ k = 1, \ldots, N. \tag{7.26}$$

As shown in the proof of Theorem 7.3.1,

$$Y_k^{(N)}(t) \ge Y_k(t), \quad k = 1, \ldots, N,\ t \ge 0. \tag{7.27}$$

Combining (7.26) and (7.27), we get:

$$Y_k(t) \le \overline{Y}_k^{(N)}(t), \quad k = 1, \ldots, N, \quad t \ge 0. \tag{7.28}$$

On the other hand, fix $\varepsilon > 0$ and $N \ge 2$. Then $\lim\limits_{M\to\infty} z_k^{(M)} = z_k$, for $k = 1, \ldots, N-1$. So there exists $M_0(N, \varepsilon)$ such that for $M > M_0(N, \varepsilon)$ we have:

$$z_1^{(M)} + \ldots + z_k^{(M)} \le z_1 + \ldots + z_k + \varepsilon, \quad k = 1, \ldots, N-1.$$

For such $M$, let $\tilde{Y} = (\tilde{Y}_1, \ldots, \tilde{Y}_N)'$, be another system of $N$ competing Brownian particles, with the same parameters and driving Brownian motions, as $Y^{(N)}$, but starting from

$(0, z_1^{(M)}, z_1^{(M)} + z_2^{(M)}, \ldots, z_1^{(M)} + z_2^{(M)} + \ldots + z_{N-1}^{(M)})'$. By Corollary 4.3.8, which corresponds to [100, Corollary 3.9],

$$\tilde{Y}_k(t) \ge \overline{Y}_k^{(M)}(t), \quad k = 1, \ldots, N, \quad t \ge 0, \tag{7.29}$$

since $\tilde{Y}$ is obtained from $\overline{Y}^{(M)}$ by removing the top $M - N$ particles. However,

$$Y^{(N)} + \varepsilon \mathbf{1}_N := (Y_1^{(N)} + \varepsilon, \ldots, Y_N^{(N)} + \varepsilon)',$$

is also a system of $N$ competing Brownian particles, with the same parameters and driving Brownian motions as $Y^{(N)}$, but starting from $(\varepsilon, z_1 + \varepsilon, \ldots, z_1 + \ldots + z_{N-1} + \varepsilon)'$. Since $Y^{(N)}(0) + \varepsilon \ge \tilde{Y}(0)$, because of (7.4.1), by Corollary 4.3.10 (i), which corresponds to [100, Corollary 3.11(i)], we have:

$$\tilde{Y}_k(t) \le Y_k^{(N)}(t) + \varepsilon, \quad k = 1, \ldots, N, \quad t \ge 0. \tag{7.30}$$

Combining (7.29) and (7.30), we get: $\overline{Y}_k^{(M)}(t) \le Y_k^{(N)}(t) + \varepsilon$, for $k = 1, \ldots, N$, and $t \ge 0$. But for every fixed $k = 1, 2, \ldots$, $\lim_{N\to\infty} Y_k^{(N)}(t) = Y_k(t)$. So there exists $N_0(k) \ge 2$ such that $Y_k^{(N_0)}(t) \le Y_k(t) + \varepsilon$. Meanwhile, for $M > M_0(N_0(k), k)$ we get:

$$\overline{Y}_k^{(M)}(t) \le Y_k(t) + 2\varepsilon. \tag{7.31}$$

We also have from (7.28) that

$$\overline{Y}_k^{(M)}(t) \ge Y_k(t). \tag{7.32}$$

Combining (7.31) and (7.32), we get (7.25). $\square$

### *7.4.2 Stationary distributions in case of skew-symmetry conditions*

In this subsection, we apply Theorem 7.4.3 to the case of the skew-symmetry condition:

$$(q_{k-1}^- + q_{k+1}^+)\sigma_k^2 = q_k^-\sigma_{k+1}^2 + q_k^+\sigma_{k-1}^2, \quad k = 2, 3, \ldots \tag{7.33}$$

Then

$$\pi^{(N)} = \bigotimes_{k=1}^{N-1} \mathrm{Exp}(\lambda_k^{(N)}), \quad \lambda_k^{(N)} = \frac{2}{\sigma_k^2 + \sigma_{k+1}^2}\left(-[R^{(N)}]^{-1}\mu^{(N)}\right)_k, \quad k = 1, \ldots, N-1.$$

Note that

$$[\pi^{(N+1)}]_{N-1} = \bigotimes_{k=1}^{N-1} \mathrm{Exp}(\lambda_k^{(N+1)}).$$

But we know that

$$[\pi^{(N+1)}]_{N-1} \preceq \pi^{(N)} = \bigotimes_{k=1}^{N-1} \mathrm{Exp}(\lambda_k^{(N)}).$$

But $\mathrm{Exp}(\lambda') \preceq \mathrm{Exp}(\lambda'')$ is equivalent to $\lambda' \geq \lambda''$. So $\lambda_k^{(N)} \leq \lambda_k^{(N+1)}$, for $k = 1, \ldots, N-1$. In other words, for every $k$, the sequence $(\lambda_k^{(N)})_{N>k}$ is nondecreasing. There exists a limit (possibly infinite)

$$\lambda_k := \lim_{N\to\infty} \lambda_k^{(N)}, \quad k = 1, 2, \ldots$$

Assume that $\lambda_k < \infty$ for all $k = 1, 2, \ldots$. Then

$$\pi = \bigotimes_{k=1}^{\infty} \mathrm{Exp}(\lambda_k). \tag{7.34}$$

If some $\lambda_k = \infty$, then we can also write (7.34), understanding that $\mathrm{Exp}(\infty) = \delta_0$ is the Dirac point mass at zero. This $\pi$ is a candidate for a stationary distribution. If the condition (7.24) is satisfied $\pi$-a.s., then it is a stationary distribution. Let us give a sufficient condition when (7.24) is satisfied or not satisfied $\pi$-a.s. (See Section 7.7 for the proof.)

**Lemma 7.4.4.** *Consider a distribution $\pi$ as in* (7.34). *Let $\Lambda_n := \sum_{k=1}^{n} \lambda_k^{-1}$.*

*(i) Let $\beta_n > 0$ be such that $\sum_{n=1}^{\infty} \beta_n^{-1} < \infty$. If*

$$\sum_{n=1}^{\infty} e^{-\alpha\Lambda_n^2 + \alpha\lambda_n^{-4}\beta_n^2} < \infty \quad \text{for all} \quad \alpha > 0,$$

*then $\pi$-a.s.* (7.24) *is satisfied. If*

$$\sum_{n=1}^{\infty} e^{-\alpha\Lambda_n^2 - \alpha\lambda_n^{-4}\beta_n^2} = \infty \quad \text{for some} \quad \alpha > 0,$$

*then it is wrong that $\pi$-a.s.* (7.24) *is satisfied.*

*(ii) If $\sup_{n\geq 1} \lambda_n < \infty$ then $\pi$-a.s.* (7.24) *is satisfied.*

*(iii) If $\sum_{n\geq 1}^{\infty} \lambda_n^{-2} < \infty$, then $\pi$-a.s.* (7.24) *is satisfied if and only if*

$$\sum_{n=1}^{\infty} e^{-\alpha\Lambda_n^2} < \infty \quad \text{for all} \quad \alpha > 0. \tag{7.35}$$

One example was already mentioned earlier:

$$\pi_\infty = \bigotimes_{n=1}^{\infty} \mathrm{Exp}(2)$$

is a stationary distribution for the infinite Atlas model, when

$$g_1 = 1,\ g_2 = g_3 = \ldots = 0,\ \sigma_1 = \sigma_2 = \ldots = 1,\ q_1^{\pm} = q_2^{\pm} = \ldots = \frac{1}{2}.$$

Indeed, the finite Atlas model of $N$ particles has stationary distribution

$$\pi^{(N)} = \bigotimes_{k=1}^{N-1} \mathrm{Exp}\left(2\frac{N-k}{N}\right)$$

for the gap process. (See [89], Example 1.) Here, for every $k = 1, 2, \ldots$

$$\lambda_k^{(N)} = 2\frac{N-k}{N} \to \lambda_k := 2 \text{ as } \ N \to \infty.$$

These $\lambda_k$, $k = 1, 2, \ldots$, satisfy Lemma 7.4.4 (ii). So $\pi_\infty$ is indeed a stationary distribution for the gap process of the infinite Atlas model. This was proved in [89], but the proof here seems to be a bit more natural.

More generally, assume the collisions are symmetric:

$$q_n^{\pm} = \frac{1}{2},\ n = 1, 2, \ldots$$

Denote, as before,

$$\overline{g}_k := \frac{1}{k}\left(g_1 + \ldots + g_k\right), \quad k = 1, 2, \ldots$$

Then the skew-symmetry condition takes the form

$$\sigma_{k+1}^2 - \sigma_k^2 = \sigma_k^2 - \sigma_{k-1}^2,\ k = 2, 3, \ldots$$

In other words, $\sigma_k^2$ must linearly depend on $k$. Because of the conditions of Theorem 7.4.3, we must have: $\sigma_k^2 = \sigma^2,\ k = 1, 2, \ldots$. In this case, $[R^{(N)}]^{-1}\mu^{(N)} < 0$ if and only if

$$\overline{g}_k > \overline{g}_N,\ k = 1, \ldots, N-1.$$

If this is true for $N > N_0$, then

$$\pi^{(N)} = \bigotimes_{k=1}^{N-1} \mathrm{Exp}\left(\lambda_k^{(N)}\right), \quad \lambda_k^{(N)} := \frac{2k}{\sigma^2}\left(\overline{g}_k - \overline{g}_N\right).$$

Suppose there exists

$$\lim_{N\to\infty} \overline{g}_N =: \overline{g}_\infty.$$

Then

$$\lambda_k^{(N)} \to \lambda_k := \frac{2k}{\sigma^2}\left(\overline{g}_k - \overline{g}_\infty\right).$$

So

$$\pi = \bigotimes_{k=1}^{\infty} \mathrm{Exp}\left(\frac{2k}{\sigma^2}\left(\overline{g}_k - \overline{g}_\infty\right)\right).$$

If $\lambda_k,\ k = 1, 2, \ldots$, satisfy Lemma 7.4.4, then $\pi$ is a stationary distribution.

*Example* 11. Consider a model with symmetric collisions, and with drift and diffusion coefficients

$$g_1 = g_2 = \ldots = g_M = 1,\ g_{M+1} = g_{M+2} = \ldots = 0,\ \sigma_1 = \sigma_2 = \ldots = 1.$$

Then

$$\overline{g}_k = k,\ k = 1, \ldots, M;\ \overline{g}_k = \frac{M}{k},\ k > M.$$

So $\overline{g}_\infty = \lim_{k\to\infty}(M/k) = 0$, and

$$\lambda_k = \begin{cases} 2k,\ 1 \le k \le M; \\ 2M,\ k > M. \end{cases}$$

Therefore,

$$\pi = \mathrm{Exp}(2) \otimes \mathrm{Exp}(4) \otimes \ldots \otimes \mathrm{Exp}(2M) \otimes \mathrm{Exp}(2M) \otimes \ldots$$

The parameters $\lambda_k, k = 1, 2, \ldots$, satisfy Lemma 7.4.4 (ii), so the conclusions of this section are valid.

### *7.4.3 Convergence Results*

Now, consider questions of *convergence* of the gap process as $t \to \infty$ to the stationary distribution $\pi$ constructed above. Let us outline the facts proved in this subsection (omitting the required conditions for now).

(i) The family of random variables $Z(t), t \ge 0$, is tight in $\mathbb{R}_+^\infty$ with respect to the metric $\rho$ from (1) (which corresponds to componentwise convergence). Any weak limit point of $Z(t)$ as $t \to \infty$ is stochastically dominated by $\pi$.

(ii) If we start the approximative version of the infinite system $Y$ with gaps stochastically larger than $\pi$, then the gap process converges weakly to $\pi$.

(iii) Any other stationary distribution for the gap process (if it exists) *must* be stochastically smaller than $\pi$.

These are generalizations of Theorem 1.4.1 from Introduction for general infinite ranked systems of competing Brownian particles in place of the infinite Atlas model. The rest of this subsection is devoted to the precise statements and proofs of these facts.

**Theorem 7.4.5.** *Consider any version (not necessarily approximative) of the infinite system of competing Brownian particles with parameters*

$$(g_n)_{n\ge1},\ (\sigma_n^2)_{n\ge1},\ (q_n^\pm)_{n\ge1}.$$

*Suppose that for $N > N_0$, we have:*

$$[R^{(N)}]^{-1}\mu^{(N)} < 0.$$

*(i) Then the family of $\mathbb{R}_+^\infty$-valued random variables $Z(t),\ t \ge 0$ is tight in $\mathbb{R}_+^\infty$ with respect to the metric $\rho$ from* (1) *(which corresponds to componentwise convergence).*

*(ii) Suppose for some sequence $t_j \uparrow \infty$ we have:*

$$Z(t_j) \ \Rightarrow\ \nu, \quad as \quad j \to \infty,$$

*where $\nu$ is some probability measure on $\mathbb{R}_+^\infty$. Then $\nu \preceq \pi$: the measure $\nu$ is stochastically dominated by $\pi$.*

*(iii) Under conditions of Theorem* 7.4.3*, every stationary distribution* $\pi'$ *for the gap process is stochastically dominated by* $\pi$*:* $\pi' \preceq \pi$*.*

*Remark* 25. Let us stress: we do not need $Y$ to be an approximative version of the system, and we do not need the initial conditions $Y(0) = y$ to satisfy (7.1).

*Proof.* (i) It suffices to show that for every $k = 1, 2, \ldots$, the family of real-valued random variables

$$Z_k = (Z_k(t), t \ge 0)$$

is tight in $\mathbb{R}_+$. Find an $N > k$ such that $[R^{(N)}]^{-1}\mu^{(N)} < 0$. Consider a finite system of $N$ competing Brownian particles with parameters $(g_n)_{1\le n\le N}$, $(\sigma_n^2)_{1\le n\le N}$, $(q_n^{\pm})_{1\le n\le N}$. Denote this system by $Y^{(N)}$, as in the proof of Theorem 7.3.1. Let $Z^{(N)} = (Z_1^{(N)}, \ldots, Z_{N-1}^{(N)})'$ be the corresponding gap process. By Proposition 3.5.1, the family of $\mathbb{R}_+^{N-1}$-valued random variables $Z^{(N)}(t), t \ge 0$, is tight in $\mathbb{R}_+^{N-1}$. By Corollary 4.3.8 and Remark 8, which correspond to [100, Corollary 3.9, Remark 7],

$$Z_k^{(N)}(t) \ge Z_k(t) \ge 0, \quad k = 1, \ldots, N-1.$$

Since the collection of real-valued random variables $Z_k^{(N)}(t)$, $t \ge 0$, is tight, then the collection $Z_k(t), t \ge 0$, is also tight.

(ii) Fix $N \ge 2$. It suffices to show that $[\nu]_{N-1} \preceq [\pi]_{N-1}$. Since $[\pi^{(M)}]_{N-1} \Rightarrow [\pi]_{N-1}$, as $M \to \infty$, it suffices to show that for $M > N$, we have: $[\nu]_{N-1} \preceq [\pi^{(M)}]_{N-1}$. Consider the system

$$Y^{(M)} = \left(Y_1^{(M)}, \ldots, Y_M^{(M)}\right)',$$

which is defined in Definition 33. Let $Z^{(M)}$ be the corresponding gap process. Then

$$Z^{(M)}(t) \;\Rightarrow\; \pi^{(M)}, \quad t \to \infty.$$

But by Corollary 4.3.8 and Remark 8, which correspond to [100, Corollary 3.9, Remark 7], $Z_k^{(M)}(t) \ge Z_k(t)$, $k = 1, \ldots, M-1$. So $[Z^{(M)}(t)]_{N-1} \ge [Z(t)]_{N-1}$, for $t \ge 0$. And $[Z(t_j)]_N \;\Rightarrow\; [\nu]_N$, as $j \to \infty$. Thus, $[\pi^{(M)}]_N \succeq [\nu]_N$.

(iii) Follows directly from (i). $\square$

**Theorem 7.4.6.** *Consider an approximative version $Y$ of the infinite system of competing Brownian particles with parameters $(g_n)_{n\ge1}$, $(\sigma_n^2)_{n\ge1}$, $(q_n^{\pm})_{n\ge1}$. Let $Z$ be the corresponding gap process. Suppose it satisfies conditions of Theorems* 7.4.3 *and* 7.4.5, *so that we can construct the distribution $\pi$. If $Z(0) \succeq \pi$, then*

$$Z(t) \Rightarrow \pi, \quad t \to \infty.$$

*Proof.* Let us show that for each $t \ge 0$ we have: $Z(t) \succeq \pi$. (Together with the results of Theorem 7.4.5, this completes the proof.) Consider another system $\overline{Y}$: an approximative version of the system with the gap process $\overline{Z}$ having stationary distribution $\pi$. Then $Z(0) \succeq \overline{Z}(0) \backsim \pi$. By Corollary 7.3.6 (ii) above, $Z(t) \succeq \overline{Z}(t) \backsim \pi$, $t \ge 0$. □

## 7.5 Triple Collisions for Infinite Systems

Let us define triple and simultaneous collisions for an infinite ranked system $Y = (Y_n)_{n\ge1}$ of competing Brownian particles.

**Definition 35.** We say that a *triple collision between particles $Y_{k-1}$, $Y_k$ and $Y_{k+1}$* occurs at time $t \ge 0$ if

$$Y_{k-1}(t) = Y_k(t) = Y_{k+1}(t).$$

We say that a *simultaneous collision* occurs at time $t \ge 0$ if for some $1 \le k < l$, we have:

$$Y_k(t) = Y_{k+1}(t) \ \text{ and } \ Y_l(t) = Y_{l+1}(t).$$

A triple collision is a particular case of a simultaneous collision. For finite systems of competing Brownian particles (both classical and ranked), the question of a.s. absence of triple collisions was studied in [58], [59], [71]. A necessary and sufficient condition for a.s. absence of any triple collisions was found in Chapter 5; see also Chapter 6 for related work. This condition is also happens to be sufficient for a.s. absence of any simultaneous collisions. In general, triple collisions are undesirable, because strong existence and pathwise uniqueness for classical systems of competing Brownian particles was shown in [59] only up to the first moment of a triple collision. Some results about triple collisions for infinite classical systems were obtained in the paper [59]. Here, we strengthen them a bit and also prove results for asymmetric collisions.

It turns out that the same necessary and sufficient condition works for infinite systems as well as for finite systems.

**Theorem 7.5.1.** *Consider the approximative version of an infinite ranked system of competing Brownian particles $Y = (Y_n)_{n\ge 1}$ with parameters*

$$(g_n)_{n\ge1},\ (\sigma_n^2)_{n\ge1},\ (q_n^{\pm})_{n\ge1}.$$

*Suppose the conditions of Theorem* 7.3.1 *hold true.*

*(i) Assume that*

$$(q_{k-1}^- + q_{k+1}^+)\sigma_k^2 \ge q_k^-\sigma_{k+1}^2 + q_k^+\sigma_{k-1}^2,\quad k = 2, 3, \ldots \tag{7.36}$$

*Then a.s. for any $t > 0$ there are no triple and no simultaneous collisions at time $t$.*

*(ii) If the condition* (7.36) *is violated for some $k = 2, 3, \ldots$, then with positive probability there exists a moment $t > 0$ such that there is a triple collision between particles with ranks $k-1,\ k,$ and $k+1$ at time $t$.*

*Proof.* The proof resembles that of Lemma 7.3.5 and uses Lemma 7.3.4.

(i) Suppose

$$D = \{\exists t > 0:\ \exists k < l:\ Y_k(t) = Y_{k+1}(t),\ Y_l(t) = Y_{l+1}(t)\}.$$

Also, let

$$D_{k,l} = \{\exists t > 0:\ Y_k(t) = Y_{k+1}(t),\ Y_l(t) = Y_{l+1}(t)\}.$$

Then

$$D = \bigcup_{k<l} D_{k,l}.$$

Suppose $\omega \in D_{k,l}$, and take the $t = t(\omega) > 0$ such that $Y_k(t) = Y_{k+1}(t)$, and $Y_l(t) = Y_{l+1}(t)$. There exists an $m > l$ such that $Y_l(t) = Y_{l+1}(t) = \ldots = Y_m(t)$, because otherwise we have a contradiction with Lemma 7.3.4 (i). Then there exist rational $q_-, q_+$ such that

$$t \in [q_-, q_+], \quad \text{and} \quad Y_m(s) < Y_{m+1}(s) \quad \text{for} \quad s \in [q_-, q_+].$$

Therefore, $L_{(m,m+1)}(t) = \text{const}$ on $[q_-, q_+]$, and, as in Lemma 7.3.5,

$$\left((Y_1(s+q_-), \ldots, Y_m(s+q_-))', 0 \le s \le q_+ - q_-\right)$$

is a ranked system of $m$ competing Brownian particles with drift coefficients $(g_k)_{1\le k\le m}$, diffusion coefficients $(\sigma_k^2)_{1\le k\le m}$, and parameters of collision $(q_k^{\pm})_{1\le k\le m}$. This system experiences a simultaneous collision at time $s = t - q_- \in (0, q_+ - q_-)$. By the Theorem 5.1.3 from Chapter 5, this event has probability zero. Let us write this formally. Let

$$\begin{aligned} D_{k,l,q_-,q_+,m} = \{\exists t \in (q_-, q_+):\ & Y_k(t) = Y_{k+1}(t),\ Y_l(t) = \ldots = Y_m(t) < Y_{m+1}(t), \\ & \text{and} \quad Y_m(s) < Y_{m+1}(s) \quad \text{for} \quad s \in (q_-, q_+)\}. \end{aligned}$$

Then

$$D = \bigcup_{k<l} D_{k,l} = \bigcup D_{k,l,q_-,q_+,m},$$

where the latter union is taken over all positive integers $k < l < m$ and positive rational numbers $q_- < q_+$. This union is countable, and by Theorem 5.1.3 from Chapter 5, which corresponds to [103, Theorem 1.2], $\mathbf{P}(D_{k,l,q_-,q_+,m}) = 0$, for each choice of $k, l, m, q_-, q_+$. Therefore, $\mathbf{P}(D) = 0$, which completes the proof of (i).

(ii) Let $B_1, B_2, \ldots$ be the driving Brownian motions of the system $Y$. Consider the ranked system of three competing Brownian particles:

$$\overline{Y} = \left(\overline{Y}_{k-1}, \overline{Y}_k, \overline{Y}_{k+1}\right)',$$

with drift coefficients $g_{k-1}, g_k, g_{k+1}$, diffusion coefficients $\sigma_{k-1}^2, \sigma_k^2, \sigma_{k+1}^2$ and parameters of collision $q_{k-1}^{\pm}, q_k^{\pm}, q_{k+1}^{\pm}$, with driving Brownian motions $B_{k-1}, B_k, B_{k+1}$, starting from

$$(Y_{k-1}(0), Y_k(0), Y_{k+1}(0))'.$$

Let $(\overline{Z}_{k-1}, \overline{Z}_k)'$ be the corresponding gap process. Then by By Corollary 4.3.9 and Remark 8, which correspond to [100, Corollary 3.10, Remark 7], we get:

$$Z_{k-1}(t) \le \overline{Z}_{k-1}(t), \quad Z_k(t) \le \overline{Z}_k(t), \quad t \ge 0.$$

But by Theorem 5.1.3 from Chapter 5, see also [103, Theorem 2], with positive probability there exists $t > 0$ such that $\overline{Y}_{k-1}(t) = \overline{Y}_k(t) = \overline{Y}_{k+1}(t)$. So $\overline{Z}_{k-1}(t) = \overline{Z}_k(t) = 0$. Therefore, with positive probability there exists $t > 0$ such that $Z_{k-1}(t) = Z_k(t) = 0$, or, in other words, $Y_{k-1}(t) = Y_k(t) = Y_{k+1}(t)$. $\square$

An interesting corollary of Theorem 5.1.3 from Chapter 5 for finite systems is that if there are a.s. no triple collisions, there there are also a.s. no simultaneous collisions. This is also true for infinite systems constructed in Theorem 7.3.1.

*Remark* 26. For symmetric collisions: $q_n^{\pm} = 1/2,\ n = 1, 2, \ldots$, this result takes the following form. There are a.s. no triple collisions if and only if the sequence $(\sigma_k^2)_{k\ge1}$ is concave. In this case, there are also a.s. no simultaneous collisions. If for some $k \ge 1$ we have:

$$\sigma_{k+1}^2 < \frac{1}{2}\left(\sigma_k^2 + \sigma_{k+2}^2\right),$$

then with positive probability there exists $t > 0$ such that $Y_k(t) = Y_{k+1}(t) = Y_{k+2}(t)$.

*Remark* 27. Let us restate the main result of [59]: for a infinite classical systems of competing Brownian particles which satisfies conditions of Theorem 7.2.1, there exists a unique strong version up to the first triple collision. In particular, if the sequence of diffusion coefficients $(\sigma_k^2)_{k\ge1}$ is concave, then there exists a unique strong solution on the infinite time horizon.

*Remark* 28. Partial results of [59] for infinite classical systems of competing Brownian particles are worth mentioning: if there are a.s. no triple collisions, then $(\sigma_k^2)_{k\ge1}$ is concave; if the sequence $(0, \sigma_1^2, \sigma_2^2, \ldots)$ is concave, then there are a.s. no triple collisions. In particular, it was already shown in [59] that the infinite Atlas model, with $\sigma_1 = \sigma_2 = \ldots = 1$, a.s. does not have triple collisions.

## 7.6 Proofs of Some Statements from Chapter 7

### 7.6.1 Proof of Theorem 7.3.8

The statement (7.23) follows from Theorem 7.3.1 immediately: the only difference is that we do not have the same driving standard Brownian motions for all $Y^{(N)},\ N \ge 2$. So instead of a.s. convergence we have weak convergence. Now, let us show (7.22). Let $\mathbf{p}_t^{(N)}$ be the ranking permutation for the vector $X^{(N)}(t) \in \mathbb{R}^N$. Then for $1 \le i \le N$ we have:

$$X_i^{(N)}(t) = x_i + \int_0^t \beta_{N,i}(s)\mathrm{d}s + \int_0^t \rho_{N,i}(s)W_i(s), \quad t \ge 0, \tag{7.37}$$

where $W_{N,1}, \ldots, W_{N,N}$ are i.i.d. standard Brownian motions,

$$\beta_{N,i}(t) = \sum_{k=1}^N 1(\mathbf{p}_t^{(N)}(k) = i)g_k, \quad \text{and} \quad \rho_{N,i}(s) = \sum_{k=1}^N 1(\mathbf{p}_t^{(N)}(k) = i)\sigma_k.$$

Note that

$$\left|\beta_{N,i}(t)\right| \le \max_{k\ge 1}|g_k| =: \overline{g},$$

and

$$\left|\rho_{N,i}(t)\right| \le \max_{k\ge 1}\sigma_k =: \overline{\sigma}.$$

Fix $T > 0$. It follows from the Arzela-Ascoli criterion and Lemma 7.7.2 that the sequence $(X_i^{(N)})_{N\ge i}$ is tight in $C[0,T]$. Every subsequence has a convergent sub-subsequence $(X_i^{(N_m)})_{m\ge 1}$. The same is true for the following sequence of $C\left([0,T], \mathbb{R}^{3k}\right)$-valued random elements, for each $k \ge 1$:

$$(X_i^{(N)}, Y_i^{(N)}, W_{N,i}, i = 1, \ldots, k)_{N\ge k}. \tag{7.38}$$

By the diagonal argument, for every subsequence $(N_m)_{m\ge 1}$ there exists a sub-subsequence $(N_m')_{m\ge 1}$ such that for every $k \ge 1$, the following subsequence of (7.38)

$$(X_i^{(N_m')}, Y_i^{(N_m')}, W_{N_m',i}, i = 1, \ldots, k)_{m\ge 1}$$

converges weakly in $C\left([0,T],\mathbb{R}^{3k}\right)$. By Skorohod theorem, we can assume that the convergence is, in fact, a.s. Let

$$X_i := \lim_{m\to\infty} X_i^{(N'_m)},\quad Y_i := \lim_{m\to\infty} Y_i^{(N'_m)},\quad W_i := \lim_{m\to\infty} W_{N'_m,i},\quad i\ge 1$$

be the a.s. uniform limit on $[0,T]$. Then $Y = (Y_i)_{i\ge1}$ is an approximative version of the infinite system of competing Brownian particles with parameters $(g_n)_{n\ge1}$, $(\sigma_n^2)_{n\ge1}$, $(\overline{q}_n^{\pm} = 1/2)_{n\ge1}$. And $W_i$ are i.i.d. standard Brownian motions. It suffices to show that $X$ is a version of the infinite classical system. Then, because the subsequence $(N_m)_{m\ge1}$ is arbitrary, and because of the tightness established above, the proof will be completed.

**Lemma 7.6.1.** *For almost every $t>0$, as $m\to\infty$, we have a.s. for all $i\ge1$:*

$$\beta_{N'_m,i}(t)\to\beta_i(t) := \sum_{k=1}^{\infty} 1(Y_k(t)=X_i(t))g_k,\quad \textit{and}\quad \rho_{N'_m,i}(t)\to\rho_i(t) := \sum_{k=1}^{\infty} 1(Y_k(t)=X_i(t))\sigma_k.$$

*Proof.* Let us prove the first convergence statement; the second statement is proved similarly.

By Lemma 7.3.5 and 3.5.3, the set $\mathcal{N}$ of times $t\in[0,T]$ when the system $Y$ or a system $Y^{(N'_m)}$ for some $m\ge1$ has a tie has Lebesgue measure zero. Fix $\varepsilon>0$ and let $\mathcal{U}\supseteq\mathcal{N}$ be an open cover of $\mathcal{N}$ in $[0,T]$ with Lebesgue measure less than $\varepsilon$.

Claim: Fix $i\ge1$. Then there exists a (random) $m_0$ such that for $m\ge m_0$, we have: for all $k\ge1$,

$$\{t\in[0,T]\setminus\mathcal{U}\mid X_i(t)=Y_k(t)\}\subseteq\left\{t\in[0,T]\setminus\mathcal{U}\mid X_i^{(N'_m)}(t)=Y_k^{(N'_m)}(t)\right\}.$$

Assuming we proved this claim, let us finish the proof of Lemma 7.6.1. Indeed, we have:

$$\beta_{N'_m,i}(t)=\beta_i(t)\ \text{ and }\ \rho_{N'_m,i}(t)=\rho_i(t),\quad t\in[0,T]\setminus\mathcal{U},\quad m>m_0.$$

This proves that

$$\beta_{N'_m,i}(t)\to\beta_i(t)\ \text{ and }\ \rho_{N'_m,i}(t)\to\rho_i(t)\ \text{ for }\ t\in[0,T]\setminus\mathcal{U}\ \text{ as }\ m\to\infty.$$

Since the set $\mathcal{U}$ (which is random) can be taken with arbitrarily small Lebesgue measure, this proves Lemma 7.6.1.

Now, let us prove this claim. Assume the converse. Then there exists a sequence $(t_j)_{j\ge 1}$ in $[0,T]\subseteq\mathcal{U}$ and a sequence $(m_j)_{j\ge1}$ such that $m_j\to\infty$ and

$$X_i(t_j)=Y_k(t_j),\quad X_i^{(N'_{m_j})}(t_j)\ne Y_k^{(N'_{m_j})}(t_j).$$

Therefore, the particle with name $i$ in the system $X^{(N'_{m_j})}$ has rank other than $k$: either larger than $k$, in which case we have:

$$X_i^{(N'_{m_j})}(t_j)\ge Y_{k+1}^{(N'_{m_j})}(t_j), \tag{7.39}$$

or smaller than $k$, in which case

$$X_i^{(N'_{m_j})}(t_j)\le Y_{k-1}^{(N'_{m_j})}(t_j). \tag{7.40}$$

By the pigeonhole principle, at least one of these inequalities is true for infinitely many $j$. Without loss of generality, we can assume that (7.39) holds for infinitely many $j\ge1$; the case when (7.40) holds for infinitely many $j\ge1$ is similar. Again, without loss of generality we can assume (7.39) holds for all $j\ge1$. There exists a convergent subsequence of $(t_j)_{j\ge1}$, because $[0,T]$ is compact. Without loss of generality, we can assume $t_j\to t_0$. Since

$$X_i^{(N'_{m_j})}(t)\to X_i(t)\ \text{ and }\ Y_{k+1}^{(N'_{m_j})}(t)\to Y_{k+1}(t)$$

uniformly on $[0,T]$, we have after letting $j\to\infty$:

$$X_i(t_0)\ge Y_{k+1}(t_0).$$

But we can also let $j\to\infty$ in $X_i(t_j)=Y_k(t_j)$. We get:

$$X_i(t_0)=Y_k(t_0).$$

Thus, $Y_{k+1}(t_0)\le Y_k(t_0)$. The reverse inequality always holds true. Therefore, there is a tie at the point $t_0$. But the set $[0,T]\setminus\mathcal{U}$ is closed, so $t_0\in[0,T]\setminus\mathcal{U}$. This contradiction completes the proof of the claim above, and with it Lemma 7.6.1. $\square$

Now, let us return to the proof of Theorem 7.3.8. Fix $t \in [0, T]$. Apply Lemma 7.7.3 to show that in $L^2(\Omega, \mathcal{F}, \mathbf{P})$, we have:

$$\int_0^t \rho_{N'_m,i}(s)\mathrm{d}W_{N'_m,i}(s) \to \int_0^t \rho_i(s)\mathrm{d}W_i(s). \tag{7.41}$$

Also, by Lebesgue dominated convergence theorem we have a.s.

$$\int_0^t \beta_{N'_m,i}(s)\mathrm{d}s \to \int_0^t \beta_i(s)\mathrm{d}s. \tag{7.42}$$

Finally, we have a.s.

$$X_i^{(N'_m)}(t) = x_i + \int_0^t \beta_{N'_m,i}(s)\mathrm{d}s + \int_0^t \rho_{N'_m,i}(s)\mathrm{d}W_{N'_m,i}(s) \to X_i(t). \tag{7.43}$$

From (7.43) and (7.42) we have that

$$\int_0^t \rho_{N'_m,i}(s)\mathrm{d}W_{N'_m,i}(s) \to X_i(t) - x_i - \int_0^t \beta_i(s)\mathrm{d}s. \tag{7.44}$$

But if a sequence of random variables converges to one limit in $L^2$ and to another limit a.s., then there limits coincide a.s. Comparing (7.41) and (7.44), we get:

$$X_i(t) = x_i + \int_0^t \beta_i(s)\mathrm{d}s + \int_0^t \rho_i(s)\mathrm{d}W_i(s),$$

which is another way to write the SDE governing the infinite classical system.

### *7.6.2 Proof of Lemma* 7.3.5

Let $D = \{Y(t) \text{ has a tie}\}$. Assume $\omega \in D$, that is, the vector $Y$ has a tie:

$$Y_{k-1}(t) < Y_k(t) = Y_{k+1}(t) = \ldots = Y_l(t) < Y_{l+1}(t). \tag{7.45}$$

This tie cannot contain infinitely many particles, because this would contradict Lemma 7.3.4. Fix a rational $q \in (Y_l(t), Y_{l+1}(t))$. By continuity of $Y_l$ and $Y_{l+1}$, there exists $M \geq 1$ such that for $s \in [t - 1/M, t + 1/M]$ we have: $Y_l(s) < q < Y_{l+1}(s)$. Let

$$C(k, l, q, M) = \Big\{ Y_{k-1}(t) < Y_k(t) = Y_{k+1}(t) = \ldots = Y_l(t) < Y_{l+1}(t),$$

$$\text{and} \;\; Y_l(s) < q < Y_{l+1}(s) \;\; \text{for all} \;\; s \in \left[t - \frac{1}{M}, t + \frac{1}{M}\right]\Bigg\}.$$

We just proved that

$$\mathbf{P}\left(D \setminus \bigcup_{M=1}^{\infty}\bigcup_{q\in\mathbb{Q}}\bigcup_{k<l} C(k,l,q,M)\right) = 0. \tag{7.46}$$

Now let us show that for every $k, l, M = 1, 2, \ldots$ with $k < l$ and for every $q \in \mathbb{Q}$, we have:

$$P(D \cap C(k,l,q,M)) = 0. \tag{7.47}$$

Since the union in (7.46) is countable, this completes the proof. If the event $C(k,l,q,M)$ happened, then we have: $([Y(u+t-1/M)], 0 \le u \le 1/M)$ behaves as a system of $l$ ranked competing Brownian particles with parameters

$$(g_n)_{1\le n\le l}, \quad (\sigma_n^2)_{1\le n\le l}, \quad (q_n^{\pm})_{1\le n\le l}.$$

By Lemma 3.5.3, the probability of a tie at $t = 1/M$ is zero, which proves (7.47).

### *7.6.3 Proof of Lemma* 7.4.2.

We can rewrite the condition from (7.24) in an equivalent form:

$$\sum_{n=1}^{\infty}\exp\left(-\alpha(y_n - y_1)^2\right) < \infty \;\; \text{for all} \;\; \alpha > 0. \tag{7.48}$$

Let us show that for a nondecreasing sequence $y$, this is equivalent to

$$\sum_{n=1}^{\infty}\exp\left(-\alpha y_n^2\right) < \infty \;\; \text{for all} \;\; \alpha > 0. \tag{7.49}$$

Indeed, $\alpha(y_n - y_1)^2 \le 2\alpha y_n^2 + 2\alpha y_1^2$, so (7.49) follows from (7.48). Conversely, $\alpha y_n^2 \le 2\alpha y_1^2 + 2\alpha(y_n - y_1)^2$, so (7.48) follows from (7.49).

*7.6.4 Proof of Lemma* 7.4.4.

(i) Use the Strong Law of Large Numbers, [110, Theorem 1.4.8] for independent $z_n \backsim \mathcal{E}(\lambda_n),\ n \ge 1$. Using that $\operatorname{Var} z_n = \lambda_n^{-2}$, we have:

$$\sum_{n=1}^{\infty} \frac{\operatorname{Var} z_n}{\beta_n \lambda_n^{-2}} = \sum_{n=1}^{\infty} \frac{1}{\beta_n} < \infty.$$

For $S_n := z_1 + \ldots + z_n,\ n \ge 1$, we have: $\mathbf{E} S_n = \Lambda_n$, and a.s.

$$c_n := \frac{S_n - \Lambda_n}{\beta_n \lambda_n^{-2}} \to 0.$$

Therefore, the sequence $(c_n)_{n \ge 1}$ is a.s. bounded: $|c_n| \le c$, where $c$ is random but finite. But $S_n = \Lambda_n + c_n \beta_n \lambda_n^{-2},\ \ n \ge 1$. Therefore, $\Lambda_n^2 - c^2 \beta_n^2 \lambda_n^{-4} \le S_n^2 \le \Lambda_n^2 + c^2 \beta_n^2 \lambda_n^{-4}$. The rest is trivial.

(ii) Let $z_k' = \lambda_k \lambda^{-1} z_k \backsim \mathcal{E}(\overline{\lambda}),\ \ \overline{\lambda} = \sup_{n \ge 1} \lambda_n$. We have: $z_1 + \ldots + z_n \ge z_1' + \ldots + z_n'$. By the Law of Large Numbers, $z_1' + \ldots + z_n' = n\overline{\lambda}^{-1}(1 + o(1))$ as $n \to \infty$. Therefore,

$$\sum_{n=1}^{\infty} e^{-\alpha(z_1 + \ldots + z_n)^2} \le \sum_{n=1}^{\infty} e^{-\alpha(z_1' + \ldots + z_n')^2} \le \sum_{n=1}^{\infty} e^{-\alpha(\overline{\lambda}^{-2}(1+o(1))n^2} < \infty.$$

(iii) Recall that $\operatorname{Var} z_n = 1/(2\lambda_n^2)$. By [110, Theorem 1.4.1], we have: $S_n - \Lambda_n$ is bounded. The rest is trivial.

## 7.7 Appendix: Technical Lemmata

**Lemma 7.7.1.** *Assume that $(y_n)_{n \ge 1}$ is a sequence of real numbers such that*

$$y_n \to \infty \quad \text{and} \quad \sum_{n=1}^{\infty} e^{-\alpha y_n^2} < \infty \quad \text{for} \quad \alpha > 0.$$

*Then for every $v \in \mathbb{R}$ and $\beta > 0$ we have:*

$$\sum_{n=1}^{\infty} \Psi\left(\frac{y_n + v}{\beta}\right) < \infty.$$

*Proof.* By [24, Chapter 7, Lemma 2], we have for $v \ge 1$:

$$\Psi(v) \le \frac{1}{\sqrt{2\pi}v} e^{-v^2/2} \le \frac{1}{\sqrt{2\pi}} e^{-v^2/2}.$$

But $x_n \to \infty$ as $n \to \infty$, and so there exists $n_0$ such that for $n \ge n_0$ we have:

$$\frac{y_n + v}{\beta} \ge 1.$$

Therefore, for $n \ge n_0$, we have:

$$\Psi\left(\frac{y_n + v}{\beta}\right) \le \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2\beta^2}(y_n + v)^2\right).$$

Using an elementary inequality $(c+d)^2 \ge c^2/2 - d^2$ for all $c, d \in \mathbb{R}$, we get:

$$\frac{1}{2\beta^2}(y_n + v)^2 \ge \frac{1}{4\beta^2} y_n^2 - \frac{1}{2\beta^2} v^2.$$

Thus,

$$\sum_{n > n_0} \Psi\left(\frac{y_n + v}{\beta}\right) \le \frac{1}{\sqrt{2\pi}} \sum_{n > n_0} \exp\left(-\frac{y_n^2}{4\beta^2} + \frac{v^2}{2\beta^2}\right)$$
$$= \frac{1}{\sqrt{2\pi}} \exp\left(\frac{v^2}{2\beta^2}\right) \sum_{n > n_0} \exp\left(-\frac{y_n^2}{4\beta^2}\right) < \infty.$$

□

**Lemma 7.7.2.** *Take a sequence $(M_n)_{n\ge1}$ of continuous local martingales on $[0, T]$, such that $M_n(0) = 0$, and $\langle M_n \rangle_t$ is differentiable for all $n$, and*

$$\sup_{n,t} \frac{\mathrm{d}\langle M_n \rangle_t}{\mathrm{d}t} = C < \infty.$$

*Then $(M_n)_{n\ge1}$ is tight in $C[0, T]$.*

*Proof.* Use [73, Chapter 2, Problem 4.11] (with obvious adjustments, because the statement in this problem is for $\mathbb{R}_+$ instead of $[0, T]$). We need only to show that

$$\sup_{X \in \mathcal{G}} \mathbf{E}(M_n(t) - M_n(s))^4 \le C_0(t - s)^2 \tag{7.50}$$

for all $0 \le s \le t \le T$ and for some constant $C_0$, depending only on $C$ and $T$. By the Burkholder-Davis-Gundy inequality, see [73, Chapter 3, Theorem 3.28], for some absolute constant $C_4 > 0$ we have:

$$\mathbf{E}(M_n(t) - M_n(s))^4 \le C_4 \mathbf{E}\left(\langle M_n \rangle_t - \langle M_n \rangle_s\right)^2 \le C_4 (C^2(t-s))^2 = C_4 C^4 (t-s)^2. \quad (7.51)$$

□

**Lemma 7.7.3.** *Take a sequence $(\xi_n)_{n \ge 0}$ of adapted processes $\xi_n = (\xi_n(t), 0 \le t \le T)$, which are bounded by a universal constant: $|\xi_n(t)| \le C$ for all $t \ge 0$ and $n = 0, 1, 2, \ldots$ Take a sequence of standard Brownian motions $(W_n)_{n \ge 0}$. Assume $\xi_n \to \xi_0$ a.s. for almost all $t \in [0, T]$ as $n \to \infty$, and $W_n \to W_0$ a.s. uniformly on $[0, T]$ as $n \to \infty$. Then in $L^2(\Omega, \mathcal{F}, \mathbf{P})$, we have:*

$$\int_0^T \xi_n(t) \mathrm{d}W_n(t) \to \int_0^T \xi_0(t) \mathrm{d}W_0(t).$$

*Proof.* We can represent

$$\begin{aligned}
&\mathbf{E}\left(\int_0^t \xi_n \mathrm{d}W_n(t) - \int_0^T \xi_0(t) \mathrm{d}W_0(t)\right)^2 \\
&\le 2\mathbf{E}\left(\int_0^T (\xi_n - \xi_0) W_n(t)\right)^2 + 2\mathbf{E}\left(\int_0^T \xi_0(t) \mathrm{d}\left(W_n(t) - W_0(t)\right)\right)^2 \\
&\le= \int_0^T \mathbf{E}\left(\xi_n(t) - \xi_0(t)\right)^2 \mathrm{d}t + \mathbf{E}\int_0^T \xi_0^2(t) \mathrm{d}\langle W_n - W_0 \rangle_t.
\end{aligned}$$

Now, the first term tends to zero because of Lebesgue dominated convergence theorem (applied twice, to the time integral and the expectation). The second term: since $W_n - W_0$ is a continuous square-integrable martingale, by Burkholder-Davis-Gundy inequalities, see [73, Chapter 3, Theorem 3.28], we have:

$$\mathbf{E}\langle W_n - W_0 \rangle_T \le C_2 \mathbf{E}(W_n(T) - W_0(T))^2. \quad (7.52)$$

Here, $C_2$ is some universal constant. (Actually, we can take $C_2 = 1$, but this is not important now.) But $W_n(T) \to W_0(T)$ a.s. So to prove that

$$\mathbf{E}(W_n(T) - W_0(T))^2 \to 0, \quad (7.53)$$

we need only show that the family $((W_n(T) - W_0(T))^2)_{n\geq 1}$ is uniformly integrable. To this end, it suffices to show that

$$\sup_{n\geq 1} \mathbf{E}\left[(W_n(T) - W_0(T))^2\right]^2 = \sup_{n\geq 1} \mathbf{E}(W_n(T) - W_0(T))^4 < \infty.$$

But this is true, because $(a+b)^4 \leq 8(a^4 + b^4)$ for all $a, b \in \mathbb{R}$, and therefore

$$\mathbf{E}(W_n(T) - W_0(T))^4 \leq 8\left(\mathbf{E}W_n^4(T) + \mathbf{E}W_0^4(T)\right) = 8\left(3T^2 + 3T^2\right) = 48T^2 < \infty.$$

Therefore, from (7.52) and (7.53) we get:

$$\mathbf{E}\langle W_n - W_0\rangle_T \to 0.$$

Thus,

$$\mathbf{E}\int_0^T \xi_0^2(t)\mathrm{d}\langle W_n - W_0\rangle_t \leq C^2\mathbf{E}\langle W_n - W_0\rangle_T \to 0.$$

The rest of the proof is trivial. $\square$

# Chapter 8

## RELATED INFINITE SYSTEMS

### *8.1 Driving Functions Which Are Not Brownian Motions*

This section is devoted to infinite ranked systems of competing particles when the driving functions are not necessarily Brownian motions. We can alter the definition of competing Brownian particles a bit, allowing for a few particles at the bottom to be driven by something else rather than Brownian motions. We prove existence of such systems and state convergence properties for their gap process.

**Definition 36.** Fix $n_0 \ge 1$ and take i.i.d. standard Brownian motions $B_n,\ n > n_0$. Fix parameters

$$g_n \in \mathbb{R},\ \sigma_n > 0,\ n > n_0,$$

as well as parameters of collision $(q^{\pm}_n)_{n\ge 1}$, and take continuous functions

$$\mathcal{X}_1, \ldots, \mathcal{X}_{n_0} : \mathbb{R}_+ \to \mathbb{R},$$

such that

$$\mathcal{X}_1(0) \le \ldots \le \mathcal{X}_{n_0}(0).$$

Take initial values $y_n,\ n > n_0$, such that

$$\mathcal{X}_{n_0}(0) \le y_{n_0+1} \le y_{n_0+2} \le \ldots$$

Define

$$\mathcal{X}_n(t) = y_n + g_n t + \sigma_n B_n(t),\ n > n_0,\ t \ge 0.$$

Suppose we have an $\mathbb{R}^\infty$-valued process $Y = (Y(t), t \ge 0)$ with continuous adapted components $Y_k = (Y_k(t), t \ge 0),\ k = 1, 2, \ldots$, and real-valued continuous adapted processes

$$L_{(k,k+1)} = (L_{(k,k+1)}(t), t \ge 0),\ k = 1, 2, \ldots$$

such that the following is true:

$$Y_k(t) = \mathcal{X}_n(t) + q_k^+ L_{(k-1,k)}(t) - q_k^- L_{(k,k+1)}(t),\ k = 1, 2, \ldots,\ t \ge 0.$$

For notational convenience, as usual, we let $L_{(0,1)} \equiv 0$. Suppose the conditions (i) and (iii) of Definition 32 hold true. Then the process $Y = (Y(t), t \ge 0)$ is called a *tweaked infinite ranked system of competing Brownian particles* with *bottom driving functions* $\mathcal{X}_1, \ldots, \mathcal{X}_{n_0}$. All other terminology is the same as in Definition 32.

*Remark* 29. The *approximative version* of a tweaked system is defined similarly to the approximate version of an infinite ranked system of competing Brownian particles, see Definition 33. To define it, we need a concept of systems of competing particles (not necessarily Brownian), when driving functions are not necessarily Brownian motions, but arbitrary continuous functions. This concept is defined in [100, Section 2, Definition 1].

First, we can state an existence result. The proof is similar to Theorem 7.3.1 from Chapter 7 and is omitted.

**Theorem 8.1.1.** *Suppose that*

$$\sum_{n>n_0} e^{-\alpha y_n^2} < \infty \quad \textit{for all} \quad \alpha > 0.$$

*Suppose that*

$$\inf_{n\ge 1} g_n > -\infty, \quad \sup_{n\ge 1} \sigma_n^2 < \infty.$$

*Then the approximative version of the system from Definition* 36 *exists.*

Now, consider the limiting behavior of the gap process. We find this behavior by comparing tweaked and non-tweaked infinite systems of competing Brownian particles. We could formulate a general theorem, but it would not be very illuminating. Instead, consider a particular case. Let

$$n_0 = 1,\ g_2 = g_3 = \ldots = 0,\ \sigma_2 = \sigma_3 = \ldots = 1,\ q_1^\pm = q_2^\pm = \frac{1}{2}.$$

Then we have an infinite Atlas model with a general continuous function which drives the bottom particle. Let us call this a *tweaked infinite Atlas model* with the *bottom driving function* $\mathcal{X}_1$.

**Theorem 8.1.2.** *Consider a tweaked infinite Atlas model (not necessarily an approximative version) with the bottom driving function*

$$\mathcal{X}_1(t) = B_1(t) + f(t),$$

*where* $B_1 = (B_1(t), t \ge 0)$ *is a standard Brownian motion, independent of* $B_2, B_3, \ldots$, *and* $f : \mathbb{R}_+ \to \mathbb{R}$ *is a deterministic function such that*

$$\varliminf_{T\to\infty} \inf_{t>s\ge T} \frac{f(t) - f(s)}{t - s} =: \gamma \in (0, \infty]. \tag{8.1}$$

*Let* $Z = (Z(t), t \ge 0)$ *be the gap process.*

*(i) Suppose* $\gamma < \infty$. *If* $\nu$ *is a weak limit point of* $Z(t)$ *as* $t \to \infty$, *that is, if* $Z(t_j) \Rightarrow \nu$ *for some increasing sequence* $t_j \to \infty$, *then*

$$\nu \preceq \pi_{2\gamma} := \bigotimes_{n=1}^{\infty} \mathrm{Exp}(2\gamma).$$

*(ii) If* $\gamma = \infty$, *then* $Z(t) \Rightarrow 0$ *as* $t \to \infty$.

*Proof.* Fix $\varepsilon \in (0, \gamma)$. Then for some $T > 0$ and for all $t > s \ge T$ we have:

$$\frac{f(t) - f(s)}{t - s} \ge \gamma - \varepsilon.$$

By the memoryless property (see [100, Remark 8]) the process $Y(T + \cdot)$ is also a tweaked infinite Atlas model with the bottom driving function $\mathcal{X}_1(T + \cdot)$, and its gap process is $Z(T+\cdot)$. Since we are interested in the behavior of $Z(t)$ as $t \to \infty$, without loss of generality we can assume $T = 0$. Then for $0 \le s < t$ we have:

$$\frac{f(t) - f(s)}{t - s} \ge \gamma - \varepsilon.$$

As mentioned earlier, systems of competing particles (not necessarily Brownian) are discussed in [100]. Now, for $N \ge 2$ let $Y^{(N)} = \left(Y_1^{(N)}, \ldots, Y_N^{(N)}\right)'$ be the system of $N$ competing particles with parameters of collision $q_n^{\pm} = 1/2,\ n = 1, \ldots, N$, and with the driving function

$$\mathcal{X}^{(N)} = (\mathcal{X}_1, y_2 + B_2, \ldots, y_N + B_N)'.$$

In effect, this is what we get when we take the system $Y$ and remove all of the ranked particles from above, except the first $N$ ones: in other words, when we remove $Y_{N+1}, Y_{N+2}, \ldots$. Let $Z^{(N)}$ be the gap process for $Y^{(N)}$. By [100, Corollary 3.9, Remark 7], we get:

$$Z_k^{(N)}(t) \ge Z_k(t), \quad k = 1, \ldots, N-1,\ t \ge 0. \tag{8.2}$$

Now, let

$$\overline{Y}^{(N)} = \left(\overline{Y}_1^{(N)}, \ldots, \overline{Y}_N^{(N)}\right)'$$

be the ranked system of $N$ competing Brownian particles with symmetric collisions, with drift coefficients $\gamma - \varepsilon, 0, \ldots, 0$ and unit diffusion coefficients, with driving Brownian motions $B_1, \ldots, B_N$, starting from $\overline{Y}^{(N)}(0) = Y^{(N)}(0)$. In other words, this is the system of $N$ competing particles with symmetric collisions and with the driving function

$$\overline{\mathcal{X}}^{(N)}(t) = ((\gamma - \varepsilon)t + B_1(t), B_2(t), \ldots, B_N(t))'.$$

Let $\overline{Z}^{(N)} = \left(\overline{Z}_1^{(N)}, \ldots, \overline{Z}_{N-1}^{(N)}\right)'$ be the corresponding gap process. We shall now compare $Z^{(N)}$ and $\overline{Z}^{(N)}$. To this end, we compare

$$\mathcal{W}^{(N)} = \left(\mathcal{X}_2^{(N)} - \mathcal{X}_1^{(N)}, \mathcal{X}_3^{(N)} - \mathcal{X}_2^{(N)}, \ldots, \mathcal{X}_N^{(N)} - \mathcal{X}_{N-1}^{(N)}\right)'$$

and

$$\overline{\mathcal{W}}^{(N)} = \left(\overline{\mathcal{X}}_2^{(N)} - \overline{\mathcal{X}}_1^{(N)}, \overline{\mathcal{X}}_3^{(N)} - \overline{\mathcal{X}}_2^{(N)}, \ldots, \overline{\mathcal{X}}_N^{(N)} - \overline{\mathcal{X}}_{N-1}^{(N)}\right)'.$$

We have:

$$\overline{\mathcal{W}}_1^{(N)}(t) = B_2(t) - B_1(t) - (\gamma - \varepsilon)t,$$

$$\mathcal{W}_1^{(N)}(t) = B_2(t) - \mathcal{X}_1(t) = B_2(t) - B_1(t) - f(t).$$

Since

$$f(t) - f(s) \geq (\gamma - \varepsilon)(t - s), \quad 0 \leq s \leq t,$$

we have:

$$\mathcal{W}_1^{(N)}(t) - \mathcal{W}_1^{(N)}(s) \leq \overline{\mathcal{W}}_1^{(N)}(t) - \overline{\mathcal{W}}_1^{(N)}(s).$$

And for $k = 2, \ldots, N-1$, we have:

$$\mathcal{W}_k^{(N)}(t) - \mathcal{W}_k^{(N)}(s) = \overline{\mathcal{W}}_k^{(N)}(t) - \overline{\mathcal{W}}_k^{(N)}(s).$$

Also, $\mathcal{W}_k^{(N)}(0) = \overline{\mathcal{W}}_k^{(N)}(0)$ for $k = 1, \ldots, N-1$. By [100, Theorem 3.1], we have:

$$Z_k^{(N)}(t) \leq \overline{Z}_k^{(N)}(t),\ k = 1, \ldots, N-1,\ t \geq 0. \tag{8.3}$$

Combining (8.2) and (8.3), we get:

$$Z_k(t) \leq \overline{Z}_k^{(N)}(t), \quad k = 1, \ldots, N-1,\ t \geq 0.$$

But by Example 1,

$$\overline{Z}^{(N)}(t) \Rightarrow \bigotimes_{k=1}^{N-1} \mathcal{E}\left(2(\gamma - \varepsilon)\frac{N-k}{N}\right), \quad t \to \infty.$$

So any weak limit point $(Z_1(t), \ldots, Z_k(t))'$ is stochastically dominated by

$$\bigotimes_{j=1}^{k} \mathcal{E}\left(2(\gamma - \varepsilon)\frac{N-j}{N}\right).$$

But $N > k$ is arbitrary. As $N \to \infty$, we have:

$$2(\gamma - \varepsilon)\frac{N-j}{N} \to 2(\gamma - \varepsilon).$$

Thus,

$$\bigotimes_{j=1}^{k} \mathcal{E}\left(2(\gamma - \varepsilon)\frac{N-j}{N}\right) \Rightarrow \bigotimes_{j=1}^{k} \mathcal{E}\left(2(\gamma - \varepsilon)\right), \quad N \to \infty.$$

Therefore, any weak limit point of $(Z_1(t), \ldots, Z_k(t))'$ is dominated by

$$\bigotimes_{j=1}^{k} \mathcal{E}\left(2(\gamma - \varepsilon)\right).$$

Snce $\varepsilon > 0$ is arbirary, any weak limit point of $(Z_1(t), \ldots, Z_k(t))'$ is dominated by

$$\bigotimes_{j=1}^{k} \mathcal{E}\left(2\gamma\right).$$

This is true for all $k = 1, 2, \ldots$, which proves (i). Part (ii) is similar. □

**Corollary 8.1.3.** *Consider a tweaked infinite Atlas model with the bottom driving function*

$$\mathcal{X}_1(t) = B_1(t) + f(t),$$

*where $f : \mathbb{R}_+ \to \mathbb{R}$ is a deterministic function which is continuous and differentiable on $\mathbb{R}_+$, and*

$$\underline{\lim}_{T\to\infty} f'(T) = \gamma \in (0, \infty].$$

*Then conclusions of Theorem* 8.1.2 *hold true.*

*Example* 12. Suppose $f(t) = t^2$, that is, the bottom particle has drift $t^2$ instead of a linear drift. Then

$$Z(t) \Rightarrow 0,\ t \to \infty.$$

## *8.2 Two-sided infinite systems of competing Brownian particles*

In this subsection, we consider infinite systems of competing Brownian particles indexed by $n \in \mathbb{Z}$ instead of $n = 1, 2, \ldots$. We could not prove an existence theorem for these systems, but we show some convergence results for the gap process, assuming a copy of such system exists.

**Definition 37.** Fix parameters

$$(g_n)_{n\in\mathbb{Z}},\ (\sigma_n^2)_{n\in\mathbb{Z}},\ (q_n^{\pm})_{n\in\mathbb{Z}}$$

with

$$\sigma_n > 0,\ 0 \le q_n^{\pm} < 1,\ q_{n+1}^{+} + q_n^{-} = 1,\ n \in \mathbb{Z}.$$

Let $B_n,\ n \in \mathbb{Z}$, be i.i.d. standard Brownian motions. Take an $\mathbb{R}^{\mathbb{Z}}$-valued process $Y = (Y(t), t \ge 0)$ with continuous adapted components $Y_n = (Y_n(t), t \ge 0),\ n \in \mathbb{Z}$, and real-valued continuous adapted processes $L_{(n,n+1)} = (L_{(n,n+1)}(t), t \ge 0)$, such that:

(i) $Y_n(t) \le Y_{n+1}(t),\ n \in \mathbb{Z},\ t \ge 0$;

(ii) $Y_n(t) = Y_n(0) + g_n t + \sigma_n B_n(t) + q_n^+ L_{(n-1,n)}(t) - q_n^- L_{(n,n+1)}(t)$;

(iii) $L_{(n,n+1)}(0) = 0$, $L_{(n,n+1)}$ is nondecreasing and can increase only when $Y_n(t) = Y_{n+1}(t)$.

Then the process $Y$ is called a *two-sided infinite ranked system of competing Brownian particles* with *drift coefficients* $(g_n)_{n\in\mathbb{Z}}$, *diffusion coefficients* $(\sigma_n^2)_{n\in\mathbb{Z}}$, and *parameters of collision* $(q_n^{\pm})_{n\in\mathbb{Z}}$. We say that $Y$ *starts from* $Y(0)$ and has *driving Brownian motions* $B_n,\ n \in \mathbb{Z}$. The component $Y_n$ is called the *nth ranked particle*. The process $L_{(n,n+1)}$ is called the *local time of collision between* $Y_n$ *and* $Y_{n+1}$. The $\mathbb{R}_+^{\mathbb{Z}}$-valued process

$$Z = (Z(t), t \ge 0), \quad Z(t) = (Z_n(t))_{n\in\mathbb{Z}}, \quad Z_n(t) = Y_{n+1}(t) - Y_n(t),$$

is called the *gap process*.

Assume a copy of such two-sided system exists. For integers $M < N$, let

$$Y^{(M,N)} = \left(Y_M^{(M,N)}, \ldots, Y_N^{(M,N)}\right)'$$

be a finite system of $N - M + 1$ competing Brownian particles with parameters

$$(g_n)_{M\le n\le N},\ (\sigma_n^2)_{M\le n\le N},\ (q_n^{\pm})_{M\le n\le N},$$

with driving Brownian motions $B_M, \ldots, B_N$, starting from $(Y_M(0), \ldots, Y_N(0))'$. Let

$$Z^{(M,N)} = \left(Z_M^{(M,N)}, \ldots, Z_{N-1}^{(M,N)}\right)'$$

be the corresponding gap process. Then by [100, Corollary 3.10] we have: for $t \ge 0$ and $M \le k < N$,

$$Z_k(t) \le Z_k^{(M,N)}(t).$$

Moreover, for $M' \le M \le k < N \le N'$, $t \ge 0$, we have:

$$Z_k^{(M',N')}(t) \le Z_k^{(M,N)}(t).$$

Suppose for sufficiently small $M$ and sufficiently large $N$ (say, for $M \le M_0$ and $N \ge N_0$) the gap process $Z^{(M,N)}$ has a stationary distribution $\pi^{(M,N)}$. A necessary and sufficient condition for this is given in Proposition 3.5.1. Let

$$(\xi_M^{(M,N)}, \ldots, \xi_N^{(M,N)})'$$

be $\mathcal{F}_0$-measurable, and distributed as $\pi^{(M,N)}$.

**Lemma 8.2.1.** *For every $k \le l \in \mathbb{Z}$, the random vector*

$$\left(\xi_k^{(M,N)}, \ldots, \xi_l^{(M,N)}\right)'$$

*stochastically decreases as $M$ decreases, and as $N$ increases. More precisely, for $M' \le M \le k \le l < N \le N'$,*

$$\left(\xi_k^{(M',N')}, \ldots, \xi_l^{(M',N')}\right)' \preceq \left(\xi_k^{(M,N)}, \ldots, \xi_l^{(M,N)}\right)'.$$

The proof is similar to that of Lemma 7.4.1. Possibly by changing the probability space, we can compare $\xi_k^{(M,N)}$ a.s. instead of stochastically. Let

$$\xi_k := \lim_{\substack{M\to-\infty \\ N\to\infty}} \xi_k^{(M,N)},\ k \in \mathbb{Z}.$$

Denote by $\pi$ the distribution of the sequence $(\xi_k)_{k\in\mathbb{Z}}$. For example, if the skew-symmetry condition holds:

$$(q_{k-1}^- + q_{k+1}^+)\sigma_k^2 = q_k^-\sigma_{k+1}^2 + q_k^+\sigma_{k-1}^2, \quad k \in \mathbb{Z},$$

then

$$\pi^{(M,N)} = \bigotimes_{k=M}^{N-1} \mathrm{Exp}\left(\lambda_k^{(M,N)}\right),$$

where $\lambda_k^{(M,N)} > 0,\ k = M, \ldots, N-1$ are parameters. Since by Lemma 8.2.1 we have: for $M' \le M \le k < N \le N'$,

$$\xi_k^{(M',N')} = \mathcal{E}(\lambda_k^{(M',N')} \preceq \xi_k^{(M,N)} \backsim \mathcal{E}\left(\lambda_k^{(M,N)}\right),$$

then

$$\lambda_k^{(M',N')} \ge \lambda_k^{(M,N)}.$$

So there exists

$$\lambda_k := \lim_{\substack{M\to-\infty\\ N\to\infty}} \lambda_k^{(M,N)} \in (0,\infty],\ k \in \mathbb{Z}.$$

If for all $k \in \mathbb{Z}$ we have: $\lambda_k < \infty$, then

$$\pi = \bigotimes_{k\in\mathbb{Z}} \mathrm{Exp}(\lambda_k). \tag{8.4}$$

If for some $k$ we have: $\lambda_k = \infty$, then we can still write (8.4), assuming that $\mathrm{Exp}(\infty) = \delta_0$, the Dirac point mass at zero.

The main result of this section is as follows. (The proof of this result is similar to Theorem 7.4.3 and is omitted.)

**Theorem 8.2.2.** *Consider a two-sided infinite ranked system of competing Brownian particles, as described in Definition* 37. *Let $Z$ be its gap process. Suppose for $M \le M_0$ and $N \ge N_0$ the gap process $Z^{(M,N)}$ has a stationary distribution $\pi^{(M,N)}$, so that we can construct a distribution $\pi$, as above.*

*(i) Then the family of $\mathbb{R}_+^{\mathbb{Z}}$-valued random variables $Z(t),\ t \ge 0$, is tight in $\mathbb{R}^{\mathbb{Z}}$ in the sense of componentwise convergence, and any weak limit point of $Z(t)$ as $t \to \infty$ is stochastically dominated by $\pi$.*

*(ii) In particular, if $\pi$ is a Dirac point mass at $0 \in \mathbb{R}^{\mathbb{Z}}$, then $Z(t) \Rightarrow 0$ as $t \to \infty$.*

**Corollary 8.2.3.** *If $\pi = \delta_0$, where $0 \in \mathbb{R}^\infty$, then $Z(t) \Rightarrow 0$ as $t \to \infty$.*

*Example* 13. Consider the case of symmetric collisions: $q_n^{\pm} = 1/2,\ n \in \mathbb{Z}$, $\sigma_n = 1,\ n \in \mathbb{Z}$, and

$$g_n = \begin{cases} 0, & n > 0; \\ 1, & n \le 0. \end{cases}$$

This is similar to the infinite Atlas model, only now there are infinitely many Atlas particles with drift 1 at the bottom of the system. The skew-symmetry condition is true, and

$$\pi^{(M,N)} = \bigotimes_{k=M}^{N} \mathrm{Exp}(\lambda_k^{(M,N)}).$$

Try to find the parameters $\lambda_k^{(-N,N)}$:

$$\overline{g}_k = \frac{1}{k+N+1}\left(g_{-N} + \ldots + g_k\right) = \begin{cases} 1,\ k \le 0; \\ \frac{N+1}{k+N+1},\ k \ge 1. \end{cases}$$

In particular,

$$\overline{g}_N = \frac{N+1}{2N+1}.$$

For $k = -N, \ldots, N$,

$$\lambda_k^{(-N,N)} = \begin{cases} \frac{2(N+1+k)N}{2N+1},\ k = -N, \ldots, 0; \\ \frac{2(N-k)(N+1)}{2N+1},\ k = 1, \ldots, N. \end{cases}$$

As $N \to \infty$, we have: $\lambda_k^{(-N,N)} \to \infty$. So $\pi = \delta_0$ is a Dirac point mass at $0 \in \mathbb{R}^\infty$, and

$$Z(t) \Rightarrow 0,\ t \to \infty.$$

Note that we did not prove existence and uniqueness for this two-sided system. We just noted that if it exists, then for every copy the gap process weakly converges to 0.

## *8.3 Construction of an Infinite-Dimensional SRBM in the Orthant*

**Definition 38.** A table $M = (M_{ij})_{i,j\ge1}$ of real numbers is called an *infinite-size matrix.* It is called *banded* if each row and each column contains only finitely many nonzero elements. For every $N \ge 1$, let $[M]_N := (M_{ij})_{1\le i,j\le N}$ be the corner principal submatrix of size $N \times N$ of an infinite-size matrix $M$. An infinite-size matrix $M$ is called *completely-$\mathcal{S}$* if for every $N \ge 1$, the matrix $[M]_N$ is completely-$\mathcal{S}$. An infinite-size matrix $M = (M_{ij})_{i,j\ge1}$ is called a *$\mathcal{Z}$-matrix* if all its off-diagonal entries are nonpositive: $M_{ij} \le 0,\ i \ne j$. For $M = (M_{ij})_{i,j\ge1}$, let $M' := (M_{ji})_{i,j\ge1}$ be the *transpose of* $M$; if $M = M'$, then $M$ is called *symmetric.* For a vector $a = (a_1, a_2, \ldots)' \in \mathbb{R}^\infty$ and an integer $N \geq 1$, let $[a]_N := (a_1, \ldots, a_N)' \in \mathbb{R}^N$. We shall call $a \in \mathbb{R}^\infty$ *banded* if only finitely many of its components are nonzero. We denote the *infinite-size identity matrix* by $I_\infty := (\delta_{ij})_{i,j\geq 1}$. We shall call an infinite-size matrix $A$ *symmetric positive definite* if for all $N \geq 1$ the matrix $[A]_N$ is symmetric positive definite.

If $M_1, M_2$ are infinite-size matrices and at least one of them is banded, then we can define the matrix product $M_1M_2$ in the usual way. Similarly, if $M$ is an infinite-size matrix and $a \in \mathbb{R}^\infty$ is a vector, and at least one of them is banded, then the product $Ma$ is well-defined. In both cases, if both $M$ and $a$ are banded, then the product is also banded.

Throughout this section, we denote $S := \mathbb{R}^\infty_+$ and $S_i := \{x \in S \mid x_i = 0\},\ i \geq 1$.

**Definition 39.** Take a symmetric positive definite infinite-size matrix $A$ and a vector $\mu \in \mathbb{R}^\infty$. A (one-sided) *infinite-dimensional* $(\mathcal{F}_t)_{t\geq 0}$*-Brownian motion* $B = (B(t), t \geq 0)$ *with drift vector* $\mu$ *and covariance matrix* $A$ is an $\mathbb{R}^\infty$-valued process $B(t) = (B_1(t), B_2(t), \ldots)'$ such that for every $N \geq 1$, $[B]_N = ([B(t)]_N, t \geq 0)$ is an $N$-dimensional Brownian motion with drift vector $[\mu]_N$ and covariance matrix $[A]_N$.

It is easy to construct such Brownian motion from infinitely many i.i.d. copies $W_1, W_2, \ldots$ of standard Brownian motions in one dimension:

$$B_k(t) = \sum_{j=1}^{k} c_{kj} W_j(t),\ k \geq 1,\ t \geq 0,$$

where the coefficients $c_{kj}$ are to be determined: first $c_{11}$, then $c_{21}, c_{22}$, etc. Then the matrix $C = (c_{kj})$ is a “square root” of $A$.

**Definition 40.** Take an infinite-size banded reflection matrix $R$. Fix $T \geq 0$. Assume $\mathcal{X} : [0, T] \to \mathbb{R}^\infty$ is a continuous function. *A solution to the Skorohod problem in the positive orthant* $S$ *with reflection matrix* $R$ *and input function* $\mathcal{X}$ is a pair $(\mathcal{Y}, \mathcal{Z})$ of continuous functions $[0, T] \to S$ which satisfy the following conditions:

(i) for every $t \in [0, T]$ we have: $\mathcal{Z}(t) = \mathcal{X}(t) + R\mathcal{Y}(t) \in S$;

(ii) for every $i \ge 1$, the function $\mathcal{Y}_i$ is nondecreasing, $\mathcal{Y}_i(0) = 0$, and $\int_0^{\infty} \mathcal{Z}_i(t)d\mathcal{Y}_i(t) = 0$. The last equality shows that $\mathcal{Y}_i$ can increase only when $\mathcal{Z}_i = 0$, that is, when $\mathcal{Z}$ is on the face $S_i$ of the boundary $\partial S$.

A similar definition can be made for $\mathbb{R}_+$ instead of $[0, T]$.

Let us describe the parameters of an SRBM in infinite dimensions. Let $R, A$ be infinite-size matrices such that $R$ is a banded $\mathcal{M}$-matrix with ones on the main diagonal, and $A$ is symmetric positive definite. Let $\mu \in \mathbb{R}^{\infty}$. Assume we have a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0},\ \mathbf{P})$ with the filtration satisfying the usual conditions.

**Definition 41.** A continuous adapted $S$-valued process $Z = (Z(t), t \ge 0)$ is an *SRBM in the infinite-dimensional orthant $S$ with reflection matrix $R$, drift vector $\mu$ and covariance matrix $A$, starting from $x \in S$*, shortly $\mathrm{SRBM}^{\infty}(R, \mu, A)$, if there exists another $\mathbb{R}^{\infty}$-valued continuous adapted process $Y = (Y(t), t \ge 0)$ such that: $(Y, Z)$ is a solution to the Skorohod problem in the orthant $S$ with reflection matrix $R$ and input function $B$, where $B$ is an infinite-dimensional Brownian motion with drift vector $\mu$ and covariance matrix $A$.

For finite systems of competing Brownian particles, the gap process is an SRBM in a finite-dimensional orthant. Similar connection exists in infinite dimensions.

**Lemma 8.3.1.** *(i) Take an infinite ranked system of competing Brownian particles with parameters*

$$(g_n)_{n\ge 1}, (\sigma_n^2)_{n\ge 1}, (q_n^{\pm})_{n\ge 1}. \tag{8.5}$$

*Then the gap process is an SRBM in the infinite-dimensional orthant $S$ with reflection matrix*

$$R = \begin{bmatrix} 1 & -q_2^- & 0 & \dots \\ -q_2^+ & 1 & -q_3^- & \dots \\ 0 & -q_3^+ & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

*covariance matrix*

$$A = \begin{bmatrix} \sigma_1^2 + \sigma_2^2 & -\sigma_2^2 & 0 & \dots \\ -\sigma_2^2 & \sigma_2^2 + \sigma_3^2 & -\sigma_3^2 & \dots \\ 0 & -\sigma_3^2 & \sigma_3^2 + \sigma_4^2 & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

*and drift vector*

$$\mu = (g_2 - g_1, g_3 - g_2, \dots)'.$$

*(ii) Conversely, suppose we constructed a process $Z = (Z(t), t \ge 0)$, which is an SRBM in the infinite-dimensional orthant $S$ with the parameters above. Then we can construct a version $Y = (Y(t), t \ge 0)$ of an infinite system of competing Brownian particles with parameters* (8.5) *such that $Z$ is the gap process for $Y$.*

*Proof.* (i) is straightforward.

(ii) By definition, $Z$ satisfies the equation: for every $k = 1, 2, \dots$

$$Z_k(t) = z_k + (g_{k+1} - g_k)\, t + B_k(t) + L_k(t) - q_k^- L_{k-1}(t) - q_{k+1}^+ L_{k+1}(t).$$

Here, $L_k$ is a continuous real-valued nondecreasing process with $L_k(0) = 0$ which can increase only when $Z_k = 0$. But we can construct the process $B = (B_k)_{k \ge 1}$ as follows:

$$B_k(t) = \sigma_{k+1} W_{k+1}(t) - \sigma_k W_k(t), \quad t \ge 0, \quad k = 1, 2, \dots$$

where $W_1, W_2, \dots$ are i.i.d. standard Brownian motions. Let

$$y_k = z_1 + \dots + z_k,\ k = 0, 1, 2, \dots,$$

and define

$$Y_k(t) = y_k + g_k t + \sigma_k W_k(t) + q_k^+ L_{k-1}(t) - q_k^- L_k(t), \quad t \ge 0, \quad k = 1, 2, \dots$$

It is easy to check that $Y$ is a required infinite system and $Z$ is the gap process for $Y$. $\square$

We can apply the Harrison-Reiman technique from [51] to prove strong existence and pathwise uniqueness.

**Theorem 8.3.2.** *Assume $\mu_n$ and $a_{nn}$ grow polynomially with respect to $n$. Let*

$$R = \begin{bmatrix} 1 & -q_2^- & 0 & \dots \\ -q_2^+ & 1 & -q_3^- & \dots \\ 0 & -q_3^+ & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

*where $q_k^{\pm} > 0,\ q_k^- + q_{k+1}^+ = 1$, and $q_k^+ \ge q > 1/2$ for all $k \ge 1$. Then $\mathrm{SRBM}^{\infty}(R, \mu, A)$ exists and is unique in the strong sense, regardless of the initial conditions.*

*Proof.* Take a sequence $\alpha = (\alpha_n)_{n\ge1}$ is a sequence of positive numbers. Consider the space

$$\Lambda_\alpha = \left\{ x = (x_n)_{n\ge1} \in \mathbb{R}^\infty \mid \|x\|_\alpha := \sum_{n=1}^{\infty} \alpha_n |x_n| < \infty \right\}.$$

This is a Banach space with the norm $\|\cdot\|_\alpha$.

**Lemma 8.3.3.** *Let*

$$B = (B_k)_{k\ge1},\ B_k = (B_k(t), t \ge 0),\ k \ge 1,$$

*be i.i.d. standard one-dimensional Brownian motions. Consider an infinite-size banded matrix $\Sigma$ such that for every $N \ge 1$, the matrix $[\Sigma]_N$ is nonsingular. Take $\mu \in \mathbb{R}^\infty$. Consider the process*

$$X = (X(t), t \ge 0), \quad X(t) = \mu t + \Sigma B(t), \quad t \ge 0.$$

*This process is $\Lambda_\alpha$-valued if*

$$\sum_{n=1}^{\infty} \alpha_n \left( \sqrt{a_{nn}} + |\mu_n| \right) < \infty, \tag{8.6}$$

*where $A = \Sigma\Sigma^T = (a_{ij})_{i,j\ge1}$.*

*Proof.* Note that

$$X_n(t) = \mu_n t + \sqrt{a_{nn}} W_n(t),\ n \ge 1,\ t \ge 0,$$

where $W_n = (W_n(t), t \ge 0),\ n \ge 1$, are standard one-dimensional Brownian motions (not necessarily independent). So

$$\mathbf{E}\|X(t)\|_\alpha \le \sum_{n=1}^{\infty} \alpha_n \left( |\mu_n| t + \sqrt{a_{nn} t} \right) < \infty.$$

$\square$

**Definition 42.** A linear mapping $F : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is called *a contraction of* $\Lambda_\alpha$ if it maps $\Lambda_\alpha$ into itself and for every $x \in \Lambda_\alpha$, we have:

$$\|F(x)\|_\alpha \le k\|x\|_\alpha,$$

where $k \in (0,1)$ is a constant.

Take a probability space on which infinitely many i.i.d. standard Brownian motions are defined.

**Lemma 8.3.4.** *Consider an* $\mathrm{SRBM}^\infty(R, \mu, \Sigma)$ *with* $A = \Sigma\Sigma^T$*, where* $R, \Sigma$ *are infinite-size banded matrices such that* $R$ *has units on the main diagonal and nonpositive elements outside it, and for every* $N \ge 1$ $[R]_N$ *is a nonsingular* $\mathcal{M}$*-matrix and* $[\Sigma]_N$ *is nonsingular. Assume there exists a sequence* $\alpha = (\alpha_n)_{n\ge1}$ *of positive real numbers such that* (8.6) *holds and the mapping*

$$x \mapsto (I_\infty - R^T)x$$

*is a contraction of* $\Lambda_\alpha$*. Then for every initial condition* $x \in S$ *(even for* $x = 0$*!), there exists a unique strong version of an* $\mathrm{SRBM}^\infty(R, \mu, \Sigma)$*.*

The proof emulates the proof of [51, Theorem 1] and is therefore omitted.

It is natural to try to find $\alpha_n = \lambda^n$, where

$$\lambda = \sqrt{\frac{1-q}{q}} \in (0,1),\ \ \alpha_n = \lambda^n,\ \ k = 2\sqrt{q(1-q)}.$$

This completes the proof of Theorem 8.3.2. $\square$

# BIBLIOGRAPHY


[1] Richard Arratia. The motion of a tagged particle in the simple symmetric exclusion system on **Z**. *Ann. Probab.*, 11(2):362–373, 1983.

[2] Adrian D. Banner, Robert Fernholz, Tomoyuki Ichiba, Ioannis Karatzas, and Vassilios Papathanakos. Hybrid atlas models. *Ann. Appl. Probab.*, 21(2):609–644, 2011.

[3] Adrian D. Banner, Robert Fernholz, and Ioannis Karatzas. Atlas models of equity markets. *Ann. Appl. Probab.*, 15(4):2296–2330, 2005.

[4] Adrian D. Banner and Raouf Ghomrasni. Local times of ranked continuous semimartingales. *Stochastic Process. Appl.*, 118(7):1244–1253, 2008.

[5] Yu. Baryshnikov. GUEs and queues. *Probab. Theory Related Fields*, 119(2):256–274, 2001.

[6] R. F. Bass and É. Pardoux. Uniqueness for diffusions with piecewise constant coefficients. *Probab. Theory Related Fields*, 76(4):557–572, 1987.

[7] Andrei N. Borodin and Paavo Salminen. *Handbook of Brownian motion—facts and formulae*. Probability and its Applications. Birkhäuser Verlag, Basel, second edition, 2002.

[8] Maury Bramson, J. G. Dai, and J. Michael Harrison. Positive recurrence of reflecting Brownian motion in three dimensions. *Ann. Appl. Probab.*, 20(2):753–783, 2010.

[9] Amarjit Budhiraja and Chihoon Lee. Long time asymptotics for constrained diffusions in polyhedral domains. *Stochastic Process. Appl.*, 117(8):1014–1036, 2007.

[10] Emmanuel Cépa and Dominique Lépingle. Diffusing particles with electrostatic repulsion. *Probability Theory and Related Fields*, 107(4):429–449, 1997.

[11] Sourav Chatterjee and Soumik Pal. A phase transition behavior for Brownian motions interacting through their ranks. *Probab. Theory Related Fields*, 147(1-2):123–159, 2010.

[12] Hong Chen. A sufficient condition for the positive recurrence of a semimartingale reflecting Brownian motion in an orthant. *Ann. Appl. Probab.*, 6(3):758–765, 1996.

[13] C. M. da Fonseca and J. Petronilho. Explicit inverses of some tridiagonal matrices. *Linear Algebra Appl.*, 325(1-3):7–21, 2001.

[14] J. G. Dai and J. Michael Harrison. Reflected Brownian motion in an orthant: numerical methods for steady-state analysis. *Ann. Appl. Probab.*, 2(1):65–86, 1992.

[15] J. G. Dai and J. Michael Harrison. Reflecting Brownian motion in three dimensions: a new proof of sufficient conditions for positive recurrence. *Math. Methods Oper. Res.*, 75(2):135–147, 2012.

[16] J. G. Dai and Thomas Kurtz. Characterization of the stationary distribution for a semimartingale reflecting brownian motion in a convex polyhedron. 2003. Unpublished Manuscript.

[17] J. G. Dai and R.J. Williams. Existence and uniqueness of semimartingale reflecting Brownian motions in convex polyhedra. *Teor. Veroyatnost. i Primenen.*, 40(1):3–53, 1995.

[18] J.G. Dai. *Steady-state analysis of reflected Brownian motions: Characterization, numerical methods and queueing applications.* 1990. Thesis (Ph.D.)–Stanford University.

[19] Donald Dawson and Jean Vaillancourt. Stochastic mckean-vlasov equations. *Nonlinear Differential Equations and Applications NoDEA*, 2(2):199–229, 1995.

[20] Amir Dembo, Mykhaylo Shkolnikov, S.R. Srinivasa Varadhan, and Ofer Zeitouni. Large deviations for diffusions interacting through their ranks. 2013. Preprint. Available at arXiv:1211.5223.

[21] Amir Dembo and Li-Tseng Tsai. Equilibrium fluctuation of the atlas model. 2015. Preprint. Available at arXiv:1503.03581.

[22] Paul Dupuis and R.J. Williams. Lyapunov functions for semimartingale reflecting Brownian motions. *Ann. Probab.*, 22(2):680–702, 1994.

[23] Ahmed El Kharroubi, Abdelghani Ben Tahar, and Abdelhak Yaacoubi. On the stability of the linear Skorohod problem in an orthant. *Math. Methods Oper. Res.*, 56(2):243–258, 2002.

[24] William Feller. *An introduction to probability theory and its applications. Vol. I.* Third edition. John Wiley & Sons Inc., New York, 1968.

[25] Robert Fernholz. On the diversity of equity markets. *J. Math. Econom.*, 31(3):393–417, 1999.

[26] Robert Fernholz. Equity portfolios generated by functions of ranked market weights. *Finance Stoch.*, 5(4):469–486, 2001.

[27] Robert Fernholz. *Stochastic portfolio theory*, volume 48 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 2002. Stochastic Modelling and Applied Probability.

[28] Robert Fernholz, Robert Garvy, and John Hannon. Diversity-weighted indexing. *The Journal of Portfolio Management*, 24(2):74–82, 1998.

[29] Robert Fernholz, Tomoyuki Ichiba, and Ioannis Karatzas. A second-order stock market model. *Annals of Finance*, 9(3):439–454, 2013.

[30] Robert Fernholz, Tomoyuki Ichiba, and Ioannis Karatzas. Two Brownian particles with rank-based characteristics and skew-elastic collisions. *Stochastic Process. Appl.*, 123(8):2999–3026, 2013.

[31] Robert Fernholz and Ioannis Karatzas. Stochastic portfolio theory: an overview. *Handbook of Numerical Analysis*, 15:89–167, 2009.

[32] Robert Fernholz, Ioannis Karatzas, and Constantinos Kardaras. Diversity and relative arbitrage in equity markets. *Finance Stoch.*, 9(1):1–27, 2005.

[33] Robert Fernholz and Brian Shay. Stochastic portfolio theory and stock market equilibrium. *The Journal of Finance*, 37(2):615–624, 1982.

[34] Robert E Fernholz and Ricardo Fernholz. Capital distribution without redistribution. 2012.

[35] P. A. Ferrari. Limit theorems for tagged particles. *Markov Process. Related Fields*, 2(1):17–40, 1996. Disordered systems and statistical physics: rigorous results (Budapest, 1995).

[36] P. A. Ferrari and L. R. G. Fontes. The net output process of a system with infinitely many queues. *Ann. Appl. Probab.*, 4(4):1129–1144, 1994.

[37] P. A. Ferrari and L. R. G. Fontes. Poissonian approximation for the tagged particle in asymmetric simple exclusion. *J. Appl. Probab.*, 33(2):411–419, 1996.

[38] Patrik L. Ferrari, Herbert Spohn, and Thomas Weiss. Scaling limit for brownian motions with one-sided collisions. 2013. Preprint. Avaliable at arXiv:1306.5095.

[39] Avner Friedman. Nonattainability of a set by a diffusion process. *Trans. Amer. Math. Soc.*, 197:245–271, 1974.

[40] Avner Friedman. *Stochastic differential equations and applications*. Dover Publications, Inc., Mineola, NY, 2006. Two volumes bound as one, Reprint of the 1975 and 1976 original published in two volumes.

[41] Jürgen Gärtner. Convergence towards Burgers' equation and propagation of chaos for weakly asymmetric exclusion processes. *Stochastic Process. Appl.*, 27(2):233–260, 1988.

[42] Jürgen Gärtner. On the McKean-Vlasov limit for interacting diffusions. *Math. Nachr.*, 137:197–248, 1988.

[43] I. I. Gihman and A. V. Skorohod. *Stochastic differential equations*. Springer-Verlag, New York-Heidelberg, 1972. Translated from the Russian by Kenneth Wickwire, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 72.

[44] Jean-Paul Haddad, Ravi R. Mazumdar, and Francisco J. Piera. Pathwise comparison results for stochastic fluid networks. *Queueing Syst.*, 66(2):155–168, 2010.

[45] T. E. Harris. Diffusion with "collisions" between particles. *J. Appl. Probability*, 2:323–338, 1965.

[46] J. Michael Harrison. The heavy traffic approximation for single server queues in series. *J. Appl. Probability*, 10:613–629, 1973.

[47] J. Michael Harrison. The diffusion approximation for tandem queues in heavy traffic. *Adv. in Appl. Probab.*, 10(4):886–905, 1978.

[48] J. Michael Harrison. *Brownian motion and stochastic flow systems*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1985.

[49] J. Michael Harrison and John J. Hasenbein. Reflected Brownian motion in the quadrant: tail behavior of the stationary distribution. *Queueing Syst.*, 61(2-3):113–138, 2009.

[50] J. Michael Harrison and I. Martin Reiman. On the distribution of multidimensional reflected Brownian motion. *SIAM J. Appl. Math.*, 41(2):345–361, 1981.

[51] J. Michael Harrison and I. Martin Reiman. Reflected Brownian motion on an orthant. *Ann. Probab.*, 9(2):302–308, 1981.

[52] J. Michael Harrison and R. J. Williams. Brownian models of open queueing networks with homogeneous customer populations. *Stochastics*, 22(2):77–115, 1987.

[53] J. Michael Harrison and R. J. Williams. Multidimensional reflected Brownian motions having exponential stationary distributions. *Ann. Probab.*, 15(1):115–137, 1987.

[54] D. G. Hobson and L. C. G. Rogers. Recurrence and transience of reflecting Brownian motion in the quadrant. *Math. Proc. Cambridge Philos. Soc.*, 113(2):387–399, 1993.

[55] Roger A. Horn and Charles R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, 1990. Corrected reprint of the 1985 original.

[56] G. Y. Hu and R. F. O'Connell. Analytical inversion of symmetric tridiagonal matrices. *J. Phys. A: Math. Gen*, 29:1511–1513, 1996.

[57] Tomoyuki Ichiba. *Topics in multi-dimensional diffusion theory: Attainability, reflection, ergodicity and rankings*. ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–Columbia University.

[58] Tomoyuki Ichiba and Ioannis Karatzas. On collisions of Brownian particles. *Ann. Appl. Probab.*, 20(3):951–977, 2010.

[59] Tomoyuki Ichiba, Ioannis Karatzas, and Mykhaylo Shkolnikov. Strong solutions of stochastic equations with rank-based coefficients. *Probab. Theory Related Fields*, 156(1-2):229–248, 2013.

[60] Tomoyuki Ichiba, Soumik Pal, and Mykhaylo Shkolnikov. Convergence rates for rank-based models with applications to portfolio theory. *Probability Theory and Related Fields*, pages 1–34, 2012.

[61] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic differential equations and diffusion processes*, volume 24 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1989.

[62] Daniel Peter Johnson. *Diffusion approximation for optimal filtering of jump processes and for queueing networks*. ProQuest LLC, Ann Arbor, MI, 1983. Thesis (Ph.D.)–The University of Wisconsin - Madison.

[63] B. Jourdain. Diffusion processes associated with nonlinear evolution equations for signed measures. *Methodol. Comput. Appl. Probab.*, 2(1):69–91, 2000.

[64] B. Jourdain. Probabilistic approximation for a porous medium equation. *Stochastic Processes and their Applications*, 89(1):81 – 99, 2000.

[65] Benjamin Jourdain and Florent Malrieu. Propagation of chaos and poincar inequalities for a system of particles interacting through their cdf. *Ann. Appl. Probab.*, 18(5):1706–1736, 10 2008.

[66] Benjamin Jourdain and Julien Reygner. Capital distribution and portfolio performance for in the mean-field atlas model. 2013. Preprint. Available at arXiv:1312.5660.

[67] Benjamin Jourdain and Julien Reygner. Propagation of chaos for rank-based interacting diffusions and long time behaviour of a scalar quasilinear parabolic equation. *Stochastic Partial Differential Equations: Analysis and Computations*, 1(3):455–506, 2013.

[68] Benjamin Jourdain and Julien Reygner. The small noise limit of order-based diffusion processes. *Electron. J. Probab.*, 19, 2014.

[69] Jourdain, Benjamin. Diffusions with a nonlinear irregular drift coefficient and probabilistic interpretation of generalized burgers' equations. *ESAIM: PS*, 1:339–355, 1997.

[70] T. Kamae, U. Krengel, and G. L. O'Brien. Stochastic inequalities on partially ordered spaces. *Ann. Probability*, 5(6):899–912, 1977.

[71] Ioannis Karatzas, Soumik Pal, and Mykhaylo Shkolnikov. Systems of brownian particles with asymmetric collisions. 2012. Preprint. Available at arXiv:1210.0259v1.

[72] Ioannis Karatzas and Andrey Sarantsev. Diverse market models of competing brownian particles with splits and mergers. 2014. Preprint. Available at arXiv:1404.0748.

[73] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[74] Offer Kella. Stochastic storage networks: stationarity and the feedforward case. *J. Appl. Probab.*, 34(2):498–507, 1997.

[75] Offer Kella. Non-product form of two-dimensional fluid networks with dependent Lévy inputs. *J. Appl. Probab.*, 37(4):1117–1122, 2000.

[76] Offer Kella and S. Ramasubramanian. Asymptotic irrelevance of initial conditions for Skorohod reflection mapping on the nonnegative orthant. *Math. Oper. Res.*, 37(2):301–312, 2012.

[77] Offer Kella and Ward Whitt. A tandem fluid network with Lévy input. In *Queueing and related models*, volume 9 of *Oxford Statist. Sci. Ser.*, pages 112–128. Oxford Univ. Press, New York, 1992.

[78] Offer Kella and Ward Whitt. Stability and structural properties of stochastic storage networks. *J. Appl. Probab.*, 33(4):1169–1180, 1996.

[79] Claude Kipnis. Central limit theorems for infinite series of queues and applications to simple exclusion. *Ann. Probab.*, 14(2):397–408, 1986.

[80] Austin J. Lemoine. Networks of queues—a survey of weak convergence results. *Management Sci.*, 24(11):1175–1193, 1977/78.

[81] P.-L. Lions and A.-S. Sznitman. Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.*, 37(4):511–537, 1984.

[82] H. P. McKean, Jr. A. Skorohod's stochastic integral equation for a reflecting barrier diffusion. *J. Math. Kyoto Univ.*, 3:85–88, 1963.

[83] H. P. McKean, Jr. A class of Markov processes associated with nonlinear parabolic equations. *Proc. Nat. Acad. Sci. U.S.A.*, 56:1907–1911, 1966.

[84] H. P. McKean, Jr. Propagation of chaos for a class of non-linear parabolic equations. In *Stochastic Differential Equations (Lecture Series in Differential Equations, Session 7, Catholic Univ., 1967)*, pages 41–57. Air Force Office Sci. Res., Arlington, Va., 1967.

[85] Carl Meyer. *Matrix analysis and applied linear algebra*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. With 1 CD-ROM (Windows, Macintosh and UNIX) and a solutions manual (iv+171 pp.).

[86] Neil O'Connell and Janosch Ortmann. Product-form invariant measures for Brownian motion with drift satisfying a skew-symmetry type condition. *ALEA Lat. Am. J. Probab. Math. Stat.*, 11(1):307–329, 2014.

[87] Neil O'Connell and Marc Yor. Brownian analogues of Burke's theorem. *Stochastic Process. Appl.*, 96(2):285–304, 2001.

[88] Janosch Ortmann. *Random Matrices, Large Deviations and Reflected Brownian Motion*. 2011. Thesis (Ph.D.)–University of Warwick.

[89] Soumik Pal and Jim Pitman. One-dimensional Brownian particle systems with rank-dependent drifts. *Ann. Appl. Probab.*, 18(6):2179–2207, 2008.

[90] Soumik Pal and Mykhaylo Shkolnikov. Concentration of measure for brownian particle systems interacting through their ranks. *Ann. Appl. Probab.*, 24(4):1482–1508, 08 2014.

[91] S. Ramasubramanian. Recurrence of projections of diffusions. *Sankhy: The Indian Journal of Statistics, Series A (1961-2002)*, 45(1):pp. 20–31, 1983.

[92] S. Ramasubramanian. Hitting of submanifolds by diffusions. *Probability Theory and Related Fields*, 78(1):149–163, 1988.

[93] S. Ramasubramanian. A subsidy-surplus model and the Skorokhod problem in an orthant. *Math. Oper. Res.*, 25(3):509–538, 2000.

[94] I. Martin Reiman. *Queueing networks in heavy traffic*. 1977. Thesis (Ph.D.)–Stanford University.

[95] I. Martin Reiman. Open queueing networks in heavy traffic. *Math. Oper. Res.*, 9(3):441–458, 1984.

[96] I. Martin Reiman and R. J. Williams. A boundary property of semimartingale reflecting Brownian motions. *Probab. Theory Related Fields*, 77(1):87–97, 1988.

[97] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[98] Julien Reygner. Chaoticity of the stationary distribution of rank-based interacting diffusions. 2014. Preprint. Available at arXiv:1408.4103.

[99] Andrey Sarantsev. On a class of diverse market models. *Ann. Finance*, 10(2):291–314, 2014.

[100] Andrey Sarantsev. Comparison techniques for competing brownian particles. 2015. Preprint. Available at arXiv:1305.1653.

[101] Andrey Sarantsev. Infinite systems of competing brownian particles. 2015. Preprint. Available at arXiv:1403.4229.

[102] Andrey Sarantsev. Multiple collisions in systems of competing brownian particles. 2015. Preprint. Available at arXiv:1309.2621.

[103] Andrey Sarantsev. Triple and simultaneous collisions of competing brownian particles. 2015. Preprint. Available at arXiv:1401.6255.

[104] Timo Seppäläinen. A scaling limit for queues in series. *Ann. Appl. Probab.*, 7(4):855–872, 1997.

[105] Mykhaylo Shkolnikov. Competing particle systems evolving by interacting Lévy processes. *Ann. Appl. Probab.*, 21(5):1911–1932, 2011.

[106] Mykhaylo Shkolnikov. Large systems of diffusions interacting through their ranks. *Stochastic Process. Appl.*, 122(4):1730–1747, 2012.

[107] A. V. Skorohod. Stochastic equations for diffusion processes with a boundary. *Teor. Verojatnost. i Primenen.*, 6:287–298, 1961.

[108] A. V. Skorohod. Stochastic equations for diffusion processes with boundaries. II. *Teor. Verojatnost. i Primenen.*, 7:5–25, 1962.

[109] Rengarajan Srinivasan. Queues in series via interacting particle systems. *Math. Oper. Res.*, 18(1):39–50, 1993.

[110] Daniel W. Stroock. *Probability theory.* Cambridge University Press, Cambridge, second edition, 2011. An analytic view.

[111] Daniel W. Stroock and S. R. S. Varadhan. Diffusion processes with boundary conditions. *Comm. Pure Appl. Math.*, 24:147–225, 1971.

[112] A.-S. Sznitman. A propagation of chaos result for Burgers' equation. *Probab. Theory Relat. Fields*, 71(4):581–613, 1986.

[113] Alain-Sol Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin, 1991.

[114] Hiroshi Tanaka. Stochastic differential equations with reflecting boundary condition in convex regions. *Hiroshima Math. J.*, 9(1):163–177, 1979.

[115] L. M. Taylor and R. J. Williams. Existence and uniqueness of semimartingale reflecting Brownian motions in an orthant. *Probab. Theory Related Fields*, 96(3):283–317, 1993.

[116] S. R. S. Varadhan and R. J. Williams. Brownian motion in a wedge with oblique reflection. *Comm. Pure Appl. Math.*, 38(4):405–443, 1985.

[117] Juan Luis Vázquez. *The porous medium equation.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007. Mathematical theory.

[118] Alexander Vervuurt and Ioannis Karatzas. Diversity-weighted portfolios with negative parameter. 2015. Preprint. Available on arXiv:1504.01026.

[119] Shinzo Watanabe. On stochastic differential equations for multi-dimensional diffusion processes with boundary conditions. *J. Math. Kyoto Univ.*, 11:169–180, 1971.

[120] Shinzo Watanabe. On stochastic differential equations for multi-dimensional diffusion processes with boundary conditions. II. *J. Math. Kyoto Univ.*, 11:545–551, 1971.

[121] R. J. Williams. Recurrence classification and invariant measure for reflected Brownian motion in a wedge. *Ann. Probab.*, 13(3):758–778, 1985.

[122] R. J. Williams. Reflected Brownian motion in a wedge: semimartingale property. *Z. Wahrsch. Verw. Gebiete*, 69(2):161–176, 1985.

[123] R. J. Williams. Local time and excursions of reflected Brownian motion in a wedge. *Publ. Res. Inst. Math. Sci.*, 23(2):297–319, 1987.

[124] R. J. Williams. Reflected Brownian motion with skew symmetric data in a polyhedral domain. *Probab. Theory Related Fields*, 75(4):459–485, 1987.

[125] R. J. Williams. Semimartingale reflecting Brownian motions in the orthant. In *Stochastic networks*, volume 71 of *IMA Vol. Math. Appl.*, pages 125–137. Springer, New York, 1995.

[126] R. J. Williams. Diffusion approximations for open multiclass queueing networks: sufficient conditions involving state space collapse. *Queueing Systems Theory Appl.*, 30(1-2):27–88, 1998.

[127] R. J. Williams. An invariance principle for semimartingale reflecting Brownian motions in an orthant. *Queueing Systems Theory Appl.*, 30(1-2):5–25, 1998.

# VITA

Andrey Sarantsev is a Ph.D. student at the University of Washington, Seattle, Department of Mathematics, 2010-2015. His adviser is Soumik Pal. He was born and raised in Moscow, Russia. He completed his undergraduate degree in Mathematics at Lomonosov Moscow State University, 2005-2010. Before this, he studied in the 57th school, a top math school in Moscow, 2001-2005. He is planning to move to the University of California, Santa Barbara, to do a three-year (2015-2018) postdoc (Visiting Assistant Professor) at the Department of Applied Probability and Statistics, where he is planning to work with Jean-Pierre Fouque and Tomoyuki Ichiba.