STAT 461/661
Practice Final 2 Solution
December 19, 2018

Name _____ Student ID #_____

TA Name _____ Section Time #_____

Problem	Points	S C O R E
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (10 points) Take two independent variables $X \sim \mathrm{Exp}(2)$ and $Y \sim \mathrm{Exp}(3)$. Find $\operatorname{Var} Z$ for

$$Z = 2X - 3Y + 5.$$

Solution: We have: $\operatorname{Var} V = \lambda^{-2}$ for $V \sim \mathrm{Exp}(\lambda)$. Therefore, $\operatorname{Var} X = 1/4$ and $\operatorname{Var} Y = 1/9$. Next, because of independence of X and Y,

$$\operatorname{Var} Z = \operatorname{Var}(2X - 3Y) = \operatorname{Var}(2X) + \operatorname{Var}(-3Y) = 4 \operatorname{Var} X + 9 \operatorname{Var} Y = 4\frac{1}{4} + 9\frac{1}{9} = \boxed{2}$$

2. (10 points) Take independent random variables $X, Y \sim \text{Uni}[1,3]$. Find the moment generating function of

$$X - 2Y + 1.$$

Solution: First, let us find the moment generating function $F_X(t)$ of X. This random variable X has density

$$p(x) = \begin{cases} \frac{1}{3-1} = \frac{1}{2}, \ 1 \le x \le 3; \\ 0, \ \text{elsewhere.} \end{cases}$$

Therefore,

$$F_X(t) = \mathbf{E}e^{tX} = \int_1^3 \frac{1}{2} e^{tx} \mathrm{d}x = \frac{1}{2} \left. \frac{e^{tx}}{t} \right|_{x=1}^{x=3} = \frac{e^{3t} - e^t}{2t}.$$

Because Y has the same distribution as X, they also have the same moment generating function:

$$F_X(t) = \mathbf{E}e^{tX} = F_Y(t) = \mathbf{E}e^{tY}.$$

Therefore, the moment generating function of $X - 2Y + 1$ is equal to

$$\mathbf{E}e^{t(X-2Y+1)} = \mathbf{E}\left[e^{tX} e^{-2tY} e^t\right] = e^t \mathbf{E}e^{tX} \mathbf{E}e^{-2tY} =$$

$$= e^t F_X(t) F_Y(-2t) = \boxed{e^t \frac{e^{3t} - e^t}{2t} \cdot \frac{e^{-6t} - e^{-2t}}{-4t}}$$

3. (10 points) Two random variables X and Y have joint density

$$\begin{cases} \frac{1}{21}(x+y), & 1 \le x \le 3, \quad 0 \le y \le 3; \\ 0, \text{otherwise.} \end{cases}$$

Find the probability $\mathbf{P}(X \ge 2,\ Y \le 2)$.

Solution: We have:

$$\mathbf{P}(X \ge 2,\ Y \le 2) = \int_2^3 \mathrm{d}x \int_0^2 \mathrm{d}y\, \frac{x+y}{21} = \int_2^3 \mathrm{d}x \frac{1}{21} \left[xy + \frac{y^2}{2}\right]\Big|_{y=0}^{y=2} =$$

$$= \frac{1}{21}\int_2^3 (2x+2)\mathrm{d}x = \frac{1}{21}\left(x^2+2x\right)\Big|_{x=2}^{x=3} = \frac{1}{21}(5+2) = \boxed{\frac{1}{3}}$$

4. (10 points) An insurance company has $N = 90000$ clients. Each comes with a claim whose distribution has moment generating function

$$F(t) = \frac{1}{3} + \frac{4}{6 - 3t}.$$

The amounts of claims are independent. Find the value-at-risk at confidence level $\alpha = 90\%$.

Solution: We have:

$$F'(t) = \frac{4 \cdot 3}{(6 - 3t)^2} = \frac{4}{3(2 - t)^2}, \quad F''(t) = \frac{8}{3(2 - t)^3}.$$

Each claim has mean μ and variance σ^2:

$$\mu = F'(0) = \frac{4}{3 \cdot 2^2} = \frac{1}{3}, \quad \sigma^2 = F''(0) - F'(0)^2 = \frac{8}{3 \cdot 2^3} - \frac{1}{9} = \frac{2}{9}.$$

Therefore, if S is the total amount of claims, then by Central Limit Theorem

$$\frac{S - N\mu}{\sqrt{N}\sigma} \approx \mathcal{N}(0, 1).$$

The quantile corresponding to 90% is $x_{90\%} = 1.282$. Therefore, the following inequality holds with probability 90%:

$$\frac{S - N\mu}{\sqrt{N}\sigma} \leq x_{90\%} \;\Rightarrow\; S \leq N\mu + \sqrt{N}\sigma x_{90\%} = \boxed{30181.3}$$

5. (10 points) There are $N = 1000$ houses, each is on fire independently of other houses with probability $p = 0.4\%$. Find the probability that there are exactly six fires.

Solution: The random variable X can be modeled as $\text{Poi}(Np) = \text{Poi}(4)$. Therefore,

$$\mathbf{P}(X = 6) = \frac{4^6}{6!}e^{-4} = \boxed{0.104}$$

6. (10 points) Take a sequence of independent Bernoulli trials, each has probability $p = 0.4$ of success. Let X be the number of trials you need to get to your first success. Find $\mathbf{E}(2X+3)^2$.

Solution: We have: $X \sim \text{Geo}(p)$, with $p = 0.4$ and $q = 0.6$. Therefore,

$$\mathbf{E}X = \frac{1}{p} = \frac{5}{2}, \quad \text{Var}\, X = \mathbf{E}X^2 - (\mathbf{E}X)^2 = \frac{q}{p^2} = \frac{15}{4}.$$

From here, we get:

$$\mathbf{E}X^2 = \frac{15}{4} + \left(\frac{5}{2}\right)^2 = 10.$$

Thus,

$$\mathbf{E}(2X+3)^2 = \mathbf{E}\left[4X^2 + 12X + 9\right] = 4\mathbf{E}X^2 + 12\mathbf{E}X + 9 = 4 \cdot 10 + 12\frac{15}{4} + 9 = \boxed{79}$$

7. (10 points) Find the expectation of a random variable X with CDF

$$F_X(x) = \begin{cases} 0, \ x < -2; \\ 0.4, \ -2 \le x < -1; \\ 0.7, \ -1 \le x < 1; \\ 1, \ x \ge 1. \end{cases}$$

Solution: This corresponds to the distribution

$$\mathbb{P}(X = -2) = 0.4, \quad \mathbb{P}(X = -1) = 0.3, \quad \mathbb{P}(X = 1) = 0.3.$$

Thus expectation is

$$\mathbb{E}[X] = (-2) \cdot 0.4 + (-1) \cdot 0.3 + 1 \cdot 0.3 = \boxed{-0.8}$$

8. (10 points) Assume that 0.1% of people from a certain population have a germ. A test gives false positive (that is, shows a person has this germ if this person does not actually have a germ) in 20% of cases when the person does not have this germ. This test gives false negative (shows a person does not have this germ if this person actually has it) in 30% of cases when this person has this germ. Suppose you pick a random person from the population and apply this test twice. Both times it gives you positive result (that is, the test says that this person has this germ). What is the probability that this person actually has this germ?

Solution: Let $F_1 = \{\text{person has a germ}\}$, $F_2 = \{\text{person does not have this germ}\}$. Let $A = \{\text{test shows positive twice}\}$. Then

$$\mathbf{P}(F_1) = 0.001,\ \mathbf{P}(F_2) = 0.999,$$

$$\mathbf{P}(A \mid F_1) = (1-0.3)^2 = 0.7^2 = 0.49,\ \ \mathbf{P}(A \mid F_2) = 0.2^2 = 0.04.$$

Using Bayes' formula, we have:

$$\mathbf{P}(F_1 \mid A) = \frac{\mathbf{P}(A \mid F_1)\mathbf{P}(F_1)}{\mathbf{P}(A \mid F_1)\mathbf{P}(F_1) + \mathbf{P}(A \mid F_2)\mathbf{P}(F_2)} = \frac{0.001 \cdot 0.49}{0.001 \cdot 0.49 + 0.999 \cdot 0.04} = \boxed{0.0121}$$

9. (10 points) Find the distribution of $XY - Z$ for random variables X, Y, Z with joint distribution

X	Y	Z	Prob.
2	1	0	0.5
0	1	-1	0.3
1	0	-2	0.2

Solution: In the first case (which happens with probability 0.5), we have: $XY - Z = 2 \cdot 1 - 0 = 2$. In the second case (which happens with probability 0.3), $XY - Z = 1$. In the third case (which happens with probability 0.2), $XY - Z = 2$. Therefore,

$$\mathbf{P}(XY - Z = 2) = 0.5 + 0.2 = 0.7, \quad \mathbf{P}(XY - Z = 0) = 0.3.$$

10. (10 points) Pick a random number from 1 to 600. Find the probability that this number is divisible by 3 but not by 5.

Solution: There are 200 numbers among $1, \ldots, 600$ which are divisible by 3. Among them, there are 40 numbers which are divisible by 5. Therefore, there are 160 numbers which are divisible by 3 but not by 5. The probability is $160/600 = \boxed{4/15}$

Cumulative Probabilities of the Standard Normal Distribution

The table gives the probabilities $\alpha = \Phi(z)$ to the left of given z–values for the standard normal distribution.

For example, the probability that a standard normal random variable Z is less than 1.53 is found at the intersection of the 1.5 rows and the 0.03 column, thus $\Phi(1.53) = P(Z \leq 1.53) = 0.9370$. Due to symmetry it holds $\Phi(-z) = 1 - \Phi(z)$ for all z.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Quantiles of the Standard Normal Distribution

For selected probabilities α, the table shows the values of the quantiles z_α such that $\Phi(z_\alpha) = P(Z \leq z_\alpha) = \alpha$, where Z is a standard normal random variable.

The quantiles satisfy the relation $z_{1-\alpha} = -z_\alpha$.

α	0.9	0.95	0.975	0.99	0.995	0.999
z_α	1.282	1.645	1.960	2.326	2.576	3.090