STAT 461/661
Practice Final 1
December 19, 2018

Name _____ Student ID #_____

Problem	Points	S C O R E
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (10 points) Consider a random variable X with distribution

$$\mathbf{P}(X=-1)=0.4, \quad \mathbf{P}(X=0)=0.4, \quad \mathbf{P}(X=1)=0.2$$

Find its expectation and the standard deviation.

Solution: We have:

$$\mathbf{E}X = (-1)\cdot 0.4 + 0\cdot 0.4 + 1\cdot 0.2 = \boxed{-0.2}$$

$$\mathbf{E}X^2 = (-1)^2\cdot 0.4 + 0^2\cdot 0.4 + 1^2\cdot 0.2 = 0.6,$$

$$\operatorname{Var} X = \mathbf{E}X^2 - (\mathbf{E}X)^2 = 0.6 - (-0.2)^2 = 0.56.$$

Thus, the standard deviation is equal to $\boxed{\sqrt{0.56}}$

2. (10 points) Find a CDF for a random variable X with density

$$f(x) = \frac{1}{4}x^3,\ 0 \le x \le 2.$$

Solution: $F(x) = 0$ for $x \le 0$ and $F(x) = 1$ for $x \ge 2$. Next, for $0 \le x \le 2$,

$$F(x) = \int_0^x f(y)\,\mathrm{d}y = \frac{x^4}{16}.$$

Answer:

$$F(x) = \begin{cases} 0,\ x \le 0; \\ x^4/16,\ 0 \le x \le 2; \\ 1,\ x \ge 2. \end{cases}$$

3. (10 points) Take two events A and B such that

$$\mathbf{P}(A \mid B) = 0.3, \quad \mathbf{P}(B \mid A) = 0.2, \quad \mathbf{P}(A \cup B) = 0.9.$$

Find the probability of A.

Solution: Let $\mathbf{P}(A) = x$ and $\mathbf{P}(B) = y$. Then

$$0.3 = \mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{\mathbf{P}(A \cap B)}{y} \quad \Rightarrow \quad \mathbf{P}(A \cap B) = 0.3y.$$

Similarly, $\mathbf{P}(A \cap B) = 0.2x$. By the inclusion-exclusion formula,

$$0.9 = \mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) = x + y - 0.3y = x + 0.7y.$$

We have the system of equations:

$$\begin{cases} 0.3y = 0.2x \\ 0.9 = x + 0.7y \end{cases} \Rightarrow \begin{cases} x = 1.5y \\ 0.9 = 1.5y + 0.7y \end{cases} \Rightarrow \begin{cases} x = 1.5y \\ y = 0.9/2.2 = 9/22. \end{cases}$$

From here, $\mathbf{P}(A) = x = 1.5 \cdot (9/22) = \boxed{27/44}$

4. (10 points) Consider a plane with $N = 300$ people. Assume half of the plane are chicken-lovers, who choose chicken with probability 75%, and the other half is pasta-lovers, who choose chicken with probability only 30%. All choices are independent. Find the quantity of chicken you need to store to satisfy chicken-lovers with probablity 99%.

Solution: Define the following sequence of random variables:

$$X_i = \begin{cases} 1, & i\text{th client chooses chicken;} \\ 0, & i\text{th client chooses pasta.} \end{cases}$$

Assume that the first 150 people are chicken-lovers, so $X_1, \ldots, X_{150}$ are Bernoulli with $p_1 = 75\%, q_1 = 25\%$, and the last 150 people are pasta-lovers, so $X_{151}, \ldots, X_{300}$ are Bernoulli with $p_2 = 30\%$ and $q_2 = 70\%$. Therefore,

$$\mathbf{E}X_i = p_1 = 0.75, \quad \operatorname{Var} X_i = p_1 q_1 = 0.75 \cdot 0.25 = \frac{3}{16}, \quad i = 1, \ldots, 150;$$

$$\mathbf{E}X_i = p_2 = 0.3, \quad \operatorname{Var} X_i = p_2 q_2 = 0.21, \quad i = 151, \ldots, 300.$$

Let $S = X_1 + \ldots + X_{300}$ be the total number of chicken servings ordered. Then

$$\mathbf{E}S = \mathbf{E}X_1 + \ldots + \mathbf{E}X_{300} = 0.75 \cdot 150 + 0.3 \cdot 150 = 157.5,$$

and because of independence,

$$\operatorname{Var} S = \operatorname{Var} X_1 + \ldots + \operatorname{Var} X_{300} = \frac{3}{16} \cdot 150 + 0.21 \cdot 150 = 59.625.$$

By Central Limit Theorem,

$$\frac{S - \mathbf{E}S}{\sqrt{\operatorname{Var} S}} = \frac{S - 157.5}{\sqrt{59.625}} \approx \mathcal{N}(0, 1).$$

Therefore, looking the 99% quantile from the table

$$\mathbf{P}\left(\frac{S - 157.5}{\sqrt{59.625}} \leq 2.33 \right) = 99\%.$$

The following inequality is true with probability 99%:

$$S \leq 157.5 + \sqrt{59.625} \cdot 2.33 = \boxed{175.49}$$

5. (10 points) There are $N_1 = 1000$ high-risk clients, each of whom can have an accident with probability $p_1 = 0.06\%$, and $N_2 = 1000$ low-risk clients, each of whom can have an accident with probability $p_2 = 0.03\%$. All accidents occur independently. Given that the number of accidents is less than 3, find the probability that there are no accidents.

Solution: We can approximate the number of accidents X by Poisson distribution: $X \sim \text{Poi}(\lambda)$, with

$$\lambda = N_1 p_1 + N_2 p_2 = 0.9.$$

This means that

$$\mathbf{P}(X = k) = \frac{0.9^k}{k!} e^{-0.9}, \quad k = 0, 1, 2, \ldots$$

Therefore,

$$\mathbf{P}(X = 0) = e^{-0.9}, \quad \mathbf{P}(X < 3) = \mathbf{P}(X = 0) + \mathbf{P}(X = 1) + \mathbf{P}(X = 2) = \left(1 + 0.9 + \frac{0.9^2}{2}\right) e^{-0.9}.$$

Thus,

$$\mathbf{P}(X = 0 \mid X < 3) = \frac{\mathbf{P}(X = 0)}{\mathbf{P}(X < 3)} = \frac{1}{2.305} = \boxed{0.43}$$

6. (10 points) Take two independent random variables $X \sim \mathrm{Exp}(2)$ and $Y \sim \mathcal{N}(-3, 10)$. For the random variable $Z = 1 + 2X - 3Y$, find $\mathbf{E}Z$ and $\operatorname{Var} Z$.

Solution: We have:

$$\mathbf{E}X = \frac{1}{2},\ \mathbf{E}Y = -3,\ \operatorname{Var} X = \frac{1}{4},\ \operatorname{Var} Y = 10.$$

Therefore,

$$\mathbf{E}Z = 1 + 2\mathbf{E}X - 3\mathbf{E}Y = 1 + 2\frac{1}{2} - 3 \cdot (-3) = \boxed{11}$$

And because of independence of X and Y,

$$\operatorname{Var} Z = \operatorname{Var}(2X) + \operatorname{Var}(-3Y) = 4 \operatorname{Var} X + 9 \operatorname{Var} Y = \boxed{91}$$

7. (10 points) For a random variable $X \sim \text{Uni}[1, 2]$, find the density of X^2.

Solution: Since X takes values from 1 to 2, then X^2 takes values from $1^2 = 1$ to $2^2 = 4$. For $x \in [1, 4]$, we have:

$$\mathbf{P}(X^2 \leq x) = \mathbf{P}(X \leq \sqrt{x}) = \int_1^{\sqrt{x}} 1 \mathrm{d}y = \sqrt{x} - 1.$$

Differentiating over x, we have: the density of X^2 is

$$\frac{\mathrm{d}(\sqrt{x} - 1)}{\mathrm{d}x} = \boxed{\frac{1}{2\sqrt{x}},\ 1 \leq x \leq 4}$$

8. (10 points) The joint density of two random variables X and Y is given by

$$p(x,y) = c(x^2+y), \quad 0 \le x \le y, \quad 0 \le y \le 1,$$

and $p(x,y) = 0$ otherwise, where $c > 0$ is a constant. Find c.

Solution: To find c, we need to note

$$\int_0^1 \int_0^y p(x,y)\mathrm{d}x\mathrm{d}y = 1.$$

Therefore,

$$c^{-1} = \int_0^1 \int_0^y \left[x^2+y\right] \mathrm{d}x\mathrm{d}y = \int_0^1 \left[\frac{x^3}{3} + xy\right]\Big|_{x=0}^{x=y} \mathrm{d}y = \int_0^1 \left[\frac{y^3}{3} + y^2\right] \mathrm{d}y =$$

$$= \left[\frac{y^4}{12} + \frac{y^3}{3}\right]\Big|_{y=0}^{y=1} = \frac{1}{12} + \frac{1}{3} = \frac{5}{12}$$

Thus, $c = \boxed{12/5}$ Next,

$$\mathbf{E}X^3 = \int_0^1 \int_0^y cx^3 \left[x^2+y\right] \mathrm{d}x\mathrm{d}y = c\int_0^1 \int_0^y \left[x^5+x^3y\right] \mathrm{d}x\mathrm{d}y = c\int_0^1 \left[\frac{x^6}{6} + \frac{x^4}{4}y\right]\Big|_{x=0}^{x=y} \mathrm{d}y =$$

$$= c\int_0^1 \left[\frac{y^6}{6} + \frac{y^5}{4}\right] \mathrm{d}y = c\left[\frac{y^7}{42} + \frac{y^6}{24}\right]\Big|_{y=0}^{y=1} = c\left[\frac{1}{42} + \frac{1}{24}\right] = \boxed{\frac{11}{70}}$$

9. (10 points) There are 50% low-risk, 30% medium-risk, and 20% high-risk clients. Each year, a high-risk client has an accident with probability 30%, a medium-risk client has an accident with probability 20%, and a low-risk client has an accident with probability 10%. Accidents in different years occur independently. Given that a randomly chosen client did not have an accident for two years in a row, what is the probability that this is a high-risk client?

Solution: Let $F_1 = \{\text{low-risk}\}$, $F_2 = \{\text{medium-risk}\}$, $F_3 = \{\text{high-risk}\}$. Then

$$\mathbf{P}(F_1) = 0.5, \quad \mathbf{P}(F_2) = 0.3, \quad \mathbf{P}(F_3) = 0.2.$$

Also, if $A = \{\text{no accident for two years}\}$, then

$$\mathbf{P}(A \mid F_1) = (1-0.1)^2 = 0.81, \quad \mathbf{P}(A \mid F_2) = (1-0.2)^2 = 0.64, \quad \mathbf{P}(A \mid F_3) = (1-0.3)^2 = 0.49.$$

Therefore, by Bayes' formula,

$$\mathbf{P}(F_3 \mid A) = \frac{\mathbf{P}(A \mid F_3)\mathbf{P}(F_3)}{\mathbf{P}(A \mid F_1)\mathbf{P}(F_1) + \mathbf{P}(A \mid F_2)\mathbf{P}(F_2) + \mathbf{P}(A \mid F_3)\mathbf{P}(F_3)} =$$

$$= \frac{0.49 \cdot 0.2}{0.49 \cdot 0.2 + 0.64 \cdot 0.3 + 0.81 \cdot 0.5} = \boxed{0.141}$$

10. (10 points) For a random variable X with density

$$p(x) = \begin{cases} cx^{-3}, & x \geq 2; \\ 0, & \text{otherwise}, \end{cases}$$

find c and find $\mathbf{E}X$.

Solution: To find c, we note that $\int_2^\infty p(x)\mathrm{d}x = 1$. Therefore,

$$c^{-1} = \int_2^\infty x^{-3}\mathrm{d}x = \frac{x^{-2}}{-2}\bigg|_{x=2}^{x=\infty} = \frac{1}{8} \quad \Rightarrow \quad \boxed{c = 8}$$

$$\mathbf{E}X = \int_2^\infty xp(x)\mathrm{d}x = 8\int_2^\infty x^{-2}\mathrm{d}x = 8\,\frac{-1}{x}\bigg|_{x=2}^{\infty} = \boxed{4}$$

Cumulative Probabilities of the Standard Normal Distribution

The table gives the probabilities $\alpha = \Phi(z)$ to the left of given z–values for the standard normal distribution.

For example, the probability that a standard normal random variable Z is less than 1.53 is found at the intersection of the 1.5 rows and the 0.03 column, thus $\Phi(1.53) = P(Z \leq 1.53) = 0.9370$. Due to symmetry it holds $\Phi(-z) = 1 - \Phi(z)$ for all z.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Quantiles of the Standard Normal Distribution

For selected probabilities α, the table shows the values of the quantiles z_α such that $\Phi(z_\alpha) = P(Z \leq z_\alpha) = \alpha$, where Z is a standard normal random variable.

The quantiles satisfy the relation $z_{1-\alpha} = -z_\alpha$.

α	0.9	0.95	0.975	0.99	0.995	0.999
z_α	1.282	1.645	1.960	2.326	2.576	3.090