Stat 753 Final Exam

May 11, 2026

Name _____

You can have a two-sided handwritten 8.5×11in note sheet, and a calculator (any model).

This exam contains 6 problems. The Z-table is on the back.

Problem	Points	S C O R E
1	10	
2	20	
3	30	
4	30	
5	30	
6	40	
Total:	160	

1. (10 points) The white noise test for the data $X_1, \ldots, X_{64}$ showed the empirical autocorrelation function for lag 1 (that is, the correlation between $X_1, \ldots, X_{63}$ and $X_2, \ldots, X_{64}$) is -0.25. Test the white noise hypothesis using asymptotic normality of the empirical autocorrelation function, on the significance level 98%.

   Solution: Let ρ_N be the empirical autocorrelation function with lag 1. The asymptotic normality is $\sqrt{N}\rho_N \Rightarrow \mathcal{N}(0,1)$ so we need to cut 1% from the right and left tail of the standard normal distribution: ± 2.326. Then $\sqrt{N}|\rho_N| > 2.326$ leads to rejection of the white noise hypothesis. Here, $N = 64$ and $\rho_N = -0.25$, therefore $\sqrt{N}|\rho_N| = 8 \cdot 0.25 = 2$. We fail to reject this hypothesis.

2. (20 points) Model a stock index as a geometric Brownian motion

$$X(t) = X(0)e^{\mu t + \sigma W(t)},$$

where W is the standard Brownian motion, $\mu = 0.1$ is the growth rate, $\sigma^2 = 0.16$ is the volatility, and $X(0) = 1000$ is the initial price. We measure time in years.

(A) What is the probability that the index level at 20 months exceeds 2000?

(B) Find the mean of the index level at 20 months.

Solution: In years, 20 months is $t = 5/3$. Then

$$X(t) = 1000 \cdot e^{0.1 \cdot 5/3} \exp\left[0.4 \cdot W(t)\right],$$

which can be represented as

$$X(t) = 1000 e^{0.5/3} e^{0.4\sqrt{5/3}Z}, \quad Z \sim \mathcal{N}(0,1).$$

(A) Solve for Z: $X(t) > 2000$ is equivalent to

$$0.4\sqrt{5/3}Z > -0.5/3 + \ln 2 \Leftrightarrow Z > \frac{-0.5/3 + \ln 2}{0.4\sqrt{5/3}} = 1.02.$$

This has probability $1 - 0.8461 = \boxed{0.1539}$

(B) We compute the mean of X:

$$\mathbb{E}\left[1000 e^{0.5/3} e^{0.4\sqrt{5/3}Z}\right] = 1000 e^{0.5/3} \exp\left[(0.4\sqrt{5/3})^2/2\right].$$

Using the moment generating function of the standard normal distribution: $\mathbb{E}[e^{uZ}] = e^{u^2/2}$, we get:

$$1000 e^{0.5/3} e^{0.16 \cdot 5/(3 \cdot 2)} = \boxed{1000 e^{0.3}}$$

3. (30 points) Consider the arithmetic random walk $S_n = Y_1 + \ldots + Y_n$ and the corresponding geometric random walk $G_n = \exp(S_n)$ with Y_k independent identically distributed with distribution

-1	0	1
0.25	0.5	0.25

(A) Find the mean of S_n.

(B) Find the standard deviation of S_n.

(C) Find the mean of G_n.

Solution: (A) The mean of Y_n is zero, so $\mathbb{E}[S_n] = \boxed{0}$

(B) The variance of Y_n is $1^2 \cdot 0.25 + 0^2 \cdot 0.5 + 1^2 \cdot 0.25 = 0.5$, so $\mathrm{Var}(S_n) = 0.5n$ and the standard deviation is $\boxed{\sqrt{0.5n}}$

(C) The mean of e^{Y_n} is $0.25 \cdot e^1 + 0.5 \cdot e^0 + 0.25 \cdot e^{-1} = 1.27$ so $\mathbb{E}[G_n] = \boxed{1.27^n}$

4. (30 points) An insurance company has cases arriving as Poisson process with 5 per month, on average.

   (A) given that the first accident happened at 0.3 months, find the probability that the second accident happens during the second month or later.

   (B) given that there is only one accident during the first month, find the conditional probability that there is exactly one accident during the second month.

   (C) using the Central Limit Theorem, if each accident costs 1000, find the amound of money u the company needs to cover all costs in 60 years with probability 90%.

Solution: (A) The waiting time from the first to the second accident is exponential with rate 0.2, if we measure time in months. We need to find the probability that this time is greater than $0.7 = 1 - 0.3$, which is $e^{-0.7/0.2} = \boxed{e^{-3.5}}$

(B) The number of accidents during the first month is independent from the number of accidents during the second month. The latter is distributed as Poisson with mean 5. Therefore, this conditional probability is equal to the unconditional probability that there is exactly one accident during the second month, which is $5e^{-5}$.

(C) The number of accidents in 10 years is Poisson with mean 3600, therefore its variance is also 3600. We can approximate it as $X \approx \mathcal{N}(3600, 3600)$ which can be represented as $X = 3600 + 60Z$ with $Z \sim \mathcal{N}(0, 1)$. But with probability 90% we have $Z > -1.282$, and $X > 3600 - 60 \cdot 1.282$.

5. (30 points) Consider the continuous-time Markov chain modeling two events: Infected and Recovered. The average time spent in Infected before it goes to Recovered is 1/2. The average time spent in Recovered before going to Infected is 2/3. Find:

   (A) The generating matrix.

   (B) The stationary distribution.

   (C) The rate of convergence.

   Solution: (A) The exit rates from Infected and Recovered are 2 and 1.5. Therefore, the generating matrix is

$$A = \begin{bmatrix} -2 & 2 \\ 1.5 & -1.5 \end{bmatrix}$$

   (B) The stationary distribution $\pi = [\pi_I, \pi_R]$ is the solution to the matrix equation $0 = \pi A$ which gives us $2\pi_I = 1.5\pi_R$ and $\boxed{\pi_I = 3/7, \pi_R = 4/7}$

   (C) The rate of convergence can be found as the nonzero eigenvalue of A which is $\lambda = -3.5$ which gives us rate $\boxed{e^{-3.5t}}$

6. (40 points) Consider the following Bonus-Malus system. Clients are classified as Gold and Silver based on the number of their car accidents per year. If a Gold client has 2 or more accidents this year, they are moved into Silver the next year. If a Silver client has 0 accidents this year, they are moved into Gold the next year. Find:

   (A) The transition matrix.

   (B) The stationary distribution.

   (C) The rate of convergence.

   (D) Given that initially, the client was Gold, find the distribution in two years.

   Use that all clients have the number of accidents N distributed as

0	1	2	3 or more
0.5	0.25	0.125	0.125

Solution: (A) The probability of Gold moving into Silver is $\mathbb{P}(N \geq 2) = 0.25$, and the probability of Silver moving into Gold is $\mathbb{P}(N = 0) = 0.5$. Therefore, the transition matrix [assuming that the states are (Gold, Silver)] is

$$\boxed{P = \begin{bmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{bmatrix}}$$

(B) The stationary distribution $\pi = \begin{bmatrix} \pi_G & \pi_S \end{bmatrix}$ can be found as the solution to

$$\pi = \pi P \Leftrightarrow \pi_S = 0.25\pi_G + 0.5\pi_S,$$

which implies $0.5\pi_S = 0.25\pi_G$, hence $\pi_G = 2\pi_S$. And $\pi_G + \pi_S = 1$, thus

$$\boxed{\pi_G = 2/3, \quad \pi_S = 1/3}$$

(C) Convergence rate can be found using eigenvalues of P: $\lambda_1 = 1$ and $\lambda_2 = 0.25$ so the rate of convergence is $\boxed{0.25^n}$

(D) The distribution at step 1 is $[0.75, 0.25]$ and therefore the distribution at step 2 is

$$[0.75, 0.25]P = [0.75 \cdot 0.75 + 0.25 \cdot 0.5, 0.25 \cdot 0.75 + 0.5 \cdot 0.25] = \boxed{\left[\frac{11}{16}, \frac{5}{16}\right]}$$

Cumulative Probabilities of the Standard Normal Distribution

The table gives the probabilities $\alpha = \Phi(z)$ to the left of given z–values for the standard normal distribution.

For example, the probability that a standard normal random variable Z is less than 1.53 is found at the intersection of the 1.5 rows and the 0.03 column, thus $\Phi(1.53) = P(Z \leq 1.53) = 0.9370$. Due to symmetry it holds $\Phi(-z) = 1 - \Phi(z)$ for all z.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Quantiles of the Standard Normal Distribution

For selected probabilities α, the table shows the values of the quantiles z_α such that $\Phi(z_\alpha) = P(Z \leq z_\alpha) = \alpha$, where Z is a standard normal random variable.

The quantiles satisfy the relation $z_{1-\alpha} = -z_\alpha$.

α	0.9	0.95	0.975	0.99	0.995	0.999
z_α	1.282	1.645	1.960	2.326	2.576	3.090