University of Nevada, Reno
Probability Theory
STAT 462/662 Final Exam
December 18, 2023

Name _____

Your exam should consist of 8 problems and a Z-table.

You are allowed a double-sided handwritten note and a calculator.

Problem	Points	S C O R E
1	15	
2	12	
3	14	
4	10	
5	14	
6	10	
7	10	
8	15	
Total:	100	

1. (15 points) An insurance company has 2 tiers of clients: Gold and Silver. Each year it rearranges them based on their last year positions. This can be modeled by a discrete-time Markov chain on the state space $\{G, S\}$ with transition matrix

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.25 & 0.75 \end{bmatrix}$$

(a, 5 points) Find the stationary distribution for this Markov chain.

(b, 5 points) Find the rate of convergence to this stationary distribution.

(c, 5 points) For a Gold client in 2021, find the probability that he is Gold in 2023.

Solution: (a) The stationary distribution $\pi = \begin{bmatrix} p & q \end{bmatrix}$ must satisfy the equation $\pi P = \pi$. The first component is

$$0.6p + 0.25q = p \Leftrightarrow 0.4p = 0.25q \Leftrightarrow q = 1.6p.$$

Also, $p + q = 1$, since V is a probability distribution. Thus,

$$q = \frac{1.6}{2.6} = \frac{8}{13}, \quad p = \frac{1}{2.6} = \frac{5}{13}.$$

(b) Find eigenvalues of the matrix P: the characteristic polynomial is

$$\det(P - zI) = \begin{vmatrix} 0.6 - z & 0.4 \\ 0.25 & 0.75 - z \end{vmatrix} = (0.6 - z)(0.75 - z) - 0.4 \cdot 0.25 = z^2 - 1.35 * z + 0.35,$$

and its roots are $z = 1$ and $z = 0.35$. Thus the rate of convergence is 0.35^n.

(c) The distribution in 2021 is $W_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ The distribution in 2022 is

$$W_1 = W_0 P = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.25 & 0.75 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix}$$

The distribution in 2023 is

$$W_2 = W_1 P = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.25 & 0.75 \end{bmatrix} = \begin{bmatrix} 0.6 \cdot 0.6 + 0.4 \cdot 0.25 & 0.6 \cdot 0.4 + 0.4 \cdot 0.25 \end{bmatrix}$$

and its first component is 0.46. This is the probabiltiy in question.

2. (12 points) Consider a square ABCD. The continuous-time Markov chain evolves as follows: It exits with intensity 2 from each vertex and chooses with equal probability its two neighbors.

   (a, 5 points) Find the generating matrix for this Markov chain.

   (b, 2 points) Find the stationary distribution.

   (c, 5 points) Find the probability that, starting from A, it hits B before C.

   Solution: (a) Putting the states in the order $\{A, B, C, D\}$, we get:

$$A = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}$$

   (b) The stationary distribution π solves the equation $\pi A = 0$. But the matrix A is symmetric, thus $A\pi^T = 0$. We can take the uniform π: $\pi = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$

   (c) The corresponding discrete-time Markov chain has transition matrix

$$P = \begin{bmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix}$$

   Assume p is the probability of hitting B before C, starting from A. Then by symmetry p is the probability of hitting C before B, starting from D. Therefore $1-p$ is the probability of hitting B before C, starting from D. Thus

$$p = 0.5 \cdot 1 + 0.5 \cdot (1 - p) \Rightarrow 1.5p = 1 \Rightarrow p = \frac{2}{3}.$$

3. (14 points) An insurance company has claims arriving according to the Poisson process with rate 3.5 per day. Each claim is random with mean 2.3 and standard deviation 1.6.

   (a, 10 points) Find the mean and the standard deviation for the overall size of all claims arrived in a week (7 days).

   (b, 4 points) Find the probability that there are exactly 10 claims during this week.

   Solution: Week is 7 days, so $t = 7$. Denote $\lambda = 3.5$ the rate and $\mu = 2.3, \sigma = 1.6$ the mean and the standard deviation of each claim.

   (a) The mean is $\lambda t \mu = 2.3 \cdot 3 \cdot 7 = 56.35$. The variance is $\lambda t(\mu^2 + \sigma^2) = 3.5 \cdot 7 \cdot (2.3^2 + 1.6^2) = 192.325$.

   (b) The distribution of the number of claims is Poisson with parameter $3.5 \cdot 7 = 24.5$. Thus the probability that there are exactly 10 claims is

$$P = \frac{24.5^{10}}{10!} \exp(-24.5) = 0.049\%.$$

4. (10 points) A stock index price is modeled as geometric Brownian motion $S(t) = \exp(X(t))$ where X has drift $\mu = 0.12$ and diffusion $\sigma^2 = 0.64$. Time is measured in months. The growth of this stock by time t is given by $G = S(t)/S(0)$.

   (a, 5 points) What is the probability that the stock price will at least double in price during one year?

   (b, 5 points) What is the mean value of G for one year?

   Solution: The random variable

$$G = \frac{S(12)}{S(0)} = \exp(0.12 \cdot 5 + 0.8 \cdot W), \quad W \sim \mathcal{N}(0, 12).$$

   Note that $W = \sqrt{12} \cdot Z$ for standard normal Z. Thus

$$G = \exp(0.6 + 0.8\sqrt{12} \cdot Z).$$

   (a) We can rewrite

$$G = 0.6 + 0.8\sqrt{12} \cdot Z > 2 \Leftrightarrow Z > \frac{\ln(2) - 0.6}{0.8\sqrt{12}} = 0.03.$$

   This has probability $1 - 0.5120 = 0.488$.

   (b) Using the MGF of the standard normal random variable, we get:

$$\mathbb{E}[G] = \exp(0.6 + 0.5 \cdot 0.8^2\sqrt{12}^2) = \exp(0.6 + 0.64 \cdot 6) = \exp(4.44).$$

5. (14 points) Statistical analysis shows a Treasury rate is modeled as an autoregression

$$R(t) = 0.45 + 0.9R(t-1) + W(t),$$

where W are IID with mean zero and standard deviation 1.5.

(a, 5 points) Find the mean and the variance of the stationary distribution.

(b, 4 points) Assuming the distribution of W is normal, let $S(\infty)$ be the random variable which has the stationary distribution and find the probability that $S(\infty) < 0$.

(c, 5 points) Find the mean and the standard deviation of $R(5)$ if $R(0) = 1.2$.

Solution: (a) Mean μ and variance σ^2:

$$\mu = \frac{0.45}{1-0.9} = 4.5, \quad \sigma^2 = \frac{1.5^2}{1-0.9^2} = \frac{2.25}{0.19} = 11.84.$$

(b) $S(\infty) = \mu + \sigma Z = 4.5 + 3.44 \cdot Z$, and so

$$\mathbb{P}(S(\infty) < 0) = \mathbb{P}(Z < -1.31) = 1 - 0.9049 = 0.0951.$$

(c) We can express $R(5)$ as a function of previous 5 innovations:

$$\begin{aligned} R(5) = {} & 1.2 \cdot 0.9^5 + (1 + 0.9 + 0.9^2 + 0.9^3 + 0.9^4) \cdot 0.45 \\ & + W(5) + 0.9W(4) + 0.9^2 W(3) + 0.9^3 W(2) + 0.9^4 W(1). \end{aligned}$$

Since the mean of each $W(k)$ is zero, the mean of $R(5)$ is

$$\mathbb{E}[R(5)] = 1.2 \cdot 0.9^5 + (1 + 0.9 + 0.9^2 + 0.9^3 + 0.9^4) \cdot 0.45 = 2.55.$$

Since the variance of each $W(k)$ is 1.5^2 and they are independent, we get:

$$\mathrm{Var}(R(5)) = (1 + 0.9^2 + 0.9^4 + 0.9^6 + 0.9^8) \cdot 1.5^2 = 7.71.$$

The standard deviation is therefore $\sqrt{7.71} = 2.78$.

6. (10 points) A stock price moves each day randomly, either up (with probability 3/4) or down (with probability 1/4). Find the probabiity that, in a trading week (5 days), more days will be up than down. Compute up to a first decimal percentage point (for example, 33.3%).

Solution: This probability can be written as $\mathbb{P}(X \geq 3)$, where $X \sim \text{Bin}(5, 3/4)$. Its mass function is

$$f(k) = \mathbb{P}(X = k) = \binom{5}{k} \left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right)^{5-k}, \quad k = 0, \ldots, 5.$$

Thus, the probability in question is:

$$\begin{aligned} f(3) + f(4) + f(5) &= \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right)^2 \binom{5}{3} + \left(\frac{3}{4}\right)^4 \left(\frac{1}{4}\right)^1 \binom{5}{4} + \left(\frac{3}{4}\right)^5 \left(\frac{1}{4}\right)^0 \binom{5}{5} \\ &= \frac{27}{1024} \cdot 10 + \frac{81}{1024} \cdot 5 + \frac{243}{1024} = 89.6\%. \end{aligned}$$

7. (10 points) Create an example of an irreduible discrete-time Markov chain with 5 states, 2 transient, 3 recurrent, and with period 2. Explicitly write the transition matrix.

Solution: Take states 1 and 2 to be transient, and states 3, 4, 5 recurrent. State 3 forms one period class, and states 4 and 5 the other class.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

8. (15 points) A random walk starts from 0 and moves up or down 1 with equal probability.

(a, 5 points) What is the probability that in $2n$ steps, its final point is also 0?

(b, 10 points) Given that in $2n$ steps, its final point is 0, what is the probability it goes negative at some point? Simplify as much as possible.

Solution: (a) A random walk starting from 0 must make n steps up and n steps down to end at 0 after $2n$ steps. The number of ways to choose the n steps up out of the $2n$ available steps is $\binom{2n}{n}$. Thus

$$P = \binom{2n}{n} \cdot 2^{-2n}.$$

(b) But the probability that the random walk ends at 0 and crosses or touches the line $y = -1$ (which means going negative at some point) by reflection principle is the same as the probabiliy of ending at the point symmetric to 0 with respect to -1, which is -2. To end at -2, the random walk must make $n+1$ negative steps and $n-1$ positive steps. The number of ways to choose these $n+1$ steps out of $2n$ is $\binom{2n}{n+1}$. As before, each path has probability 2^{-2n}. This has probability $Q = \binom{2n}{n+1} \cdot 2^{-2n}$. Thus the conditional probability is

$$\frac{Q}{P} = \frac{(2n)!/(n+1)!(n-1)!}{(2n)!/(n!^2)} = \frac{n!/(n-1)!}{(n+1)!/n!} = \frac{n}{n+1}.$$

Cumulative Probabilities of the Standard Normal Distribution

The table gives the probabilities $\alpha = \Phi(z)$ to the left of given z–values for the standard normal distribution.

For example, the probability that a standard normal random variable Z is less than 1.53 is found at the intersection of the 1.5 rows and the 0.03 column, thus $\Phi(1.53) = P(Z \leq 1.53) = 0.9370$. Due to symmetry it holds $\Phi(-z) = 1 - \Phi(z)$ for all z.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Quantiles of the Standard Normal Distribution

For selected probabilities α, the table shows the values of the quantiles z_α such that $\Phi(z_\alpha) = P(Z \leq z_\alpha) = \alpha$, where Z is a standard normal random variable.

The quantiles satisfy the relation $z_{1-\alpha} = -z_\alpha$.

α	0.9	0.95	0.975	0.99	0.995	0.999
z_α	1.282	1.645	1.960	2.326	2.576	3.090