Stat 461/661 Final Exam
Friday, May 12, 2023

Name _____ Student ID #_____

Your exam should consist of 10 problems, followed by a Z-table. Check!

You are allowed a double-sided handwritten note and a calculator.

Problem	Points	S C O R E
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (10 points) Find the cumulative distribution function for X with density

$$f(x) = \begin{cases} 3x^{-4}, \, x \geq 1; \\ 0, \quad \text{else.} \end{cases}$$

Solution: We have $F(x) = 0$ for $x \leq 1$, because the random variable is always greater than 1. For $x \geq 1$ we get:

$$F(x) = \int_1^x f(y)\,\mathrm{d}y = \int_1^x (3y^{-4})\,\mathrm{d}y = y^{-3}\Big|_{y=1}^{y=x} = 1 - x^{-3}.$$

Thus the answer is

$$F(x) = \boxed{\begin{cases} 1 - x^{-3}, \, x \geq 1; \\ 0, \, x \leq 1. \end{cases}}$$

2. (10 points) Using moment generating functions, find $\mathbb{E}[X^3]$ and $\mathbb{E}\left[e^{X/2}\right]$ for $X \sim \Gamma(3,1)$.

Solution: For $X \sim \Gamma(3,1)$, its MGF is

$$M(t) = \mathbb{E}\left[e^{tX}\right] = \left(\frac{1}{1-t}\right)^3 = (1-t)^{-3}.$$

Thus $\mathbb{E}\left[e^{X/3}\right] = M(1/2) = \boxed{8}$ and $\mathbb{E}[X^4] = M^{(3)}(0)$. Compute the third derivative:

$$M^{(3)}(t) = (-1)(-3)(-1)(-4)(-1)(-5)(1-t)^{-6} \;\Rightarrow\; M^{(3)}(0) = \boxed{60}$$

3. (10 points) For independent $X \sim \mathcal{N}(0, 2)$ and $Y \sim \mathcal{N}(1, 3)$, find $\mathbb{P}(X - 2Y > -4)$.

Solution: The distribution of $X - 2Y$ is also normal, with mean $0 - 2 \cdot 1 = -2$ and variance $2 + (-2)^2 \cdot 3 = 14$. We can represent $X - 2Y = -2 + \sqrt{14}Z$ for $Z \sim \mathcal{N}(0, 1)$. Thus

$$\mathbb{P}(X - 2Y > -4) = \mathbb{P}(-2 + \sqrt{14}Z > -4) = \mathbb{P}(Z > -2/\sqrt{14}) = \mathbb{P}(Z < 2/\sqrt{14}).$$

Since $2/\sqrt{14} = 0.53$, from the Z-table we get: the probability is $\boxed{0.7019}$

4. (10 points) There are $N = 1000$ clients, each has an accident with independent loss Exp(2). Every client is charged the same premium x. Using the Central Limit Theorem, find an x such that the company can pay all its claims with probability 90%.

Solution: Let $X_i \sim \text{Exp}(2)$ be the loss of the ith client, $i = 1, \ldots, N$. Then $S = X_1 + \ldots + X_N$ is the total loss, which the company needs to cover, and Nx is the total amount of cash the company has. We wish to find x such that $\mathbb{P}(S \leq Nx) = 90\%$. By the Central Limit Theorem,

$$\frac{S - \mathbb{E}S}{\sqrt{\text{Var}\, S}} \sim \mathcal{N}(0, 1).$$

By properties of the exponential distribution, $\mathbb{E}[X_i] = 0.5$ and $\text{Var}(X_i) = 0.25$. Thus

$$\mathbb{E}[S] = \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_N] = 1000 \cdot 0.5 = 500,$$

and by independence of $X_1, \ldots, X_N$, we get:

$$\text{Var}(S) = \text{Var}(X_1) + \ldots + \text{Var}(X_N) = 1000 \cdot 0.25 = 250.$$

From the Z-table, we get: With probability 90%,

$$\frac{S - 500}{\sqrt{250}} \leq 1.282 \;\Leftrightarrow\; S \leq 500 + \sqrt{250} \cdot 1.282 = 520.27.$$

Thus we need $Nx = 520.27 \;\Rightarrow x = \boxed{0.52}$

5. (10 points) In the population, 10% have a germ. A medical test identifies this germ if the patient has it with probability 80%, and it gives correct result if the patient does not have this germ with probability 90%. Given that a randomly chosen person has positive result, what is the probability that he, in fact, has this germ?

Solution: Let F_1 and F_2 be the events that this patient has a germ and does not have a germ. Let A be the event that the result is positive. Then

$$\mathbb{P}(F_1) = 0.1,\ \mathbb{P}(F_2) = 0.9,\ \mathbb{P}(A \mid F_1) = 0.8,\ \mathbb{P}(A \mid F_2) = 0.1.$$

By Bayes' formula,

$$\mathbb{P}(F_1 \mid A) = \frac{\mathbb{P}(A \mid F_1) \cdot \mathbb{P}(F_1)}{\mathbb{P}(A \mid F_1) \cdot \mathbb{P}(F_1) + \mathbb{P}(A \mid F_2) \cdot \mathbb{P}(F_2)} = \frac{0.1 \cdot 0.8}{0.1 \cdot 0.8 + 0.9 \cdot 0.1} = \boxed{\frac{8}{17}}$$

6. (10 points) A small insurance company has 8 clients, each has accident with probability 20%, independently of others. Find the probability that there are at most two accidents.

Solution: The number of accidents is $X \sim \mathrm{Bin}(8, 0.2)$. Thus

$$\mathbb{P}(X = k) = \binom{8}{k} \cdot 0.2^k \cdot 0.8^{8-k}, \quad k = 0, \ldots, 8.$$

In particular, the probability $\mathbb{P}(X \leq 2)$ is equal to

$$\binom{8}{0} \cdot 0.8^8 + \binom{8}{1} \cdot 0.2 \cdot 0.8^7 + \binom{8}{2} \cdot 0.2^2 \cdot 0.8^6 = 0.1678 + 0.3355 + 0.2936 = \boxed{0.797}$$

7. (10 points) For two random variables (X, Y) with joint density

$$f(x,y) = \begin{cases} x+y, \ 0 \le x \le 1, \, 0 \le y \le 1, \\ 0, \quad \text{else}, \end{cases}$$

find the conditional density of X given $Y = 1/3$.

Solution: The marginal density of Y is given by

$$f_Y(y) = \int_0^1 f(x,y)\,\mathrm{d}x = \frac{1}{2} + y, \ 0 \le y \le 1.$$

The conditional density of X given Y is given by

$$\frac{f(x,y)}{f_Y(y)} = \frac{x+y}{y+1/2}, \ 0 \le x \le 1.$$

For $Y = 1/3$, this becomes

$$p(y) = \frac{x+1/3}{1/2+1/3} = \boxed{\frac{6}{5}x + \frac{2}{5}, \ 0 \le x \le 1}$$

8. (10 points) For the previous problem, find $\mathbb{E}[X^2Y]$.

Solution: We have:

$$\begin{aligned}
\mathbb{E}\left[X^2Y\right] &= \int_0^1 \int_0^1 x^2y(x+y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_0^1 \left[x^3y + x^2y^2\right]\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \left[\frac{x^3y^2}{2} + \frac{x^2y^3}{3}\right]\Bigg|_{y=0}^{y=1} \mathrm{d}x = \int_0^1 \left[\frac{1}{2}x^3 + \frac{1}{3}x^2\right]\,\mathrm{d}x \\
&= \left[\frac{1}{8}x^4 + \frac{1}{9}x^3\right]\Bigg|_{x=0}^{x=1} = \frac{1}{8} + \frac{1}{9} = \boxed{\frac{17}{72}}
\end{aligned}$$

9. (10 points) Take two events A and B, both having probability 0.48. The probability that neither A nor B happens is 0.1. Find $\mathbb{P}(A \mid B)$.

Solution: The probability of $A \cup B$ is $1 - 0.1 = 0.9$. By the inclusion-exclusion formula,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = 0.48 + 0.48 - 0.9 = 0.06.$$

Thus the conditional probability is given by

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{0.06}{0.48} = \frac{1}{8} = \boxed{0.125}$$

10. (10 points) Assume each Fortune 500 company defaults with probability 0.5%, independently of others. Using Poisson approximation, find the probability that there is at most one default.

Solution: We have: $X \sim \text{Poi}(\lambda)$ for $\lambda = 500 \cdot 0.5\% = 2.5$. Thus

$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \ k = 0, 1, 2, \ldots$$

$$\mathbb{P}(X \leq 1) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1) = e^{-2.5} + 2.5e^{-2.5} = 3.5e^{-2.5} = \boxed{0.287}$$

Cumulative Probabilities of the Standard Normal Distribution

The table gives the probabilities $\alpha = \Phi(z)$ to the left of given z–values for the standard normal distribution.

For example, the probability that a standard normal random variable Z is less than 1.53 is found at the intersection of the 1.5 rows and the 0.03 column, thus $\Phi(1.53) = P(Z \leq 1.53) = 0.9370$. Due to symmetry it holds $\Phi(-z) = 1 - \Phi(z)$ for all z.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Quantiles of the Standard Normal Distribution

For selected probabilities α, the table shows the values of the quantiles z_α such that $\Phi(z_\alpha) = P(Z \leq z_\alpha) = \alpha$, where Z is a standard normal random variable.

The quantiles satisfy the relation $z_{1-\alpha} = -z_\alpha$.

α	0.9	0.95	0.975	0.99	0.995	0.999
z_α	1.282	1.645	1.960	2.326	2.576	3.090