A First Course in Probability
University of Nevada, Reno
STAT 461/661 Final Exam
Tuesday, December 16

Name _____

Your exam should consist of 8 problems and a Z-table. You can use a two-sided handwritten letter-sized note and a calculator. Please show your work.

Problem	Points	S C O R E
1	10	
2	15	
3	10	
4	15	
5	20	
6	20	
7	15	
8	20	
Total:	125	

1. (10 points) An auto insurance company insures drivers of all ages. An actuary compiled the following statistics on the company's insured drivers:

Age of Driver	Probability of Accident	Portion of Insured Drivers
16–20	0.06	0.08
21–30	0.03	0.15
31–65	0.02	0.49
66–99	0.04	0.28

A randomly selected driver that the company insures has an accident. Calculate the probability that the driver was age 16-20.

Solution: Partition: $F_1 =$ age 16–20, $F_2 =$ age 21–30, $F_3 =$ age 31–65, $F_4 =$ age 66–99. Let A be the event of an accident. Then

$$\begin{aligned} \mathbb{P}(F_1) &= 6\%, \quad \mathbb{P}(A \mid F_1) = 8\%; \\ \mathbb{P}(F_2) &= 3\%, \quad \mathbb{P}(A \mid F_2) = 15\%; \\ \mathbb{P}(F_3) &= 2\%, \quad \mathbb{P}(A \mid F_3) = 49\%; \\ \mathbb{P}(F_4) &= 4\%, \quad \mathbb{P}(A \mid F_4) = 28\%. \end{aligned}$$

Applying Bayes' theorem, we get:

$$\mathbb{P}(F_1 \mid A) = \frac{0.06 \cdot 0.08}{0.06 \cdot 0.08 + 0.03 \cdot 0.15 + 0.02 \cdot 0.49 + 0.04 \cdot 0.28} = \frac{48}{303} = 15.8\%.$$

2. (15 points) A boutique insurance company has 3 clients. They have accidents independently from each other, with probabilities 20%, 40%, and 50%. Find the probability that at least two clients have accidents.

Solution: Let A_k be the event that the kth client has an accident. Two or three clients having an accident is the event C which can be represented as a disjoint union:

$$\begin{aligned} C &= B_0 \cup B_1 \cup B_2 \cup B_3; \quad \text{where} \\ B_0 &= A_1 \cap A_2 \cap A_3, \\ B_1 &= A_1^c \cap A_2 \cap A_3, \\ B_2 &= A_1 \cap A_2^c \cap A_3, \\ B_3 &= A_1 \cap A_2 \cap A_3^c. \end{aligned}$$

By independence, we have:

$$\begin{aligned} \mathbb{P}(B_0) &= 0.2 \cdot 0.4 \cdot 0.5 = 4\%, \\ \mathbb{P}(B_1) &= (1 - 0.2) \cdot 0.4 \cdot 0.5 = 16\%, \\ \mathbb{P}(B_2) &= 0.2 \cdot (1 - 0.4) \cdot 0.5 = 6\%, \\ \mathbb{P}(B_3) &= 0.2 \cdot 0.4 \cdot (1 - 0.5) = 4\%. \end{aligned}$$

Since these four events are disjoint, then

$$\mathbb{P}(C) = \mathbb{P}(B_0) + \mathbb{P}(B_1) + \mathbb{P}(B_2) + \mathbb{P}(B_3) = 30\%.$$

Alternatively, we can represent C as a non-disjoint union of three events:

$$A_1 \cap A_2, \quad A_1 \cap A_3, \quad A_2 \cap A_3.$$

Uniting these, we count the triple intersection B_0 thrice, so we need to subtract it twice:

$$\mathbb{P}(C) = \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_1 \cap A_3) + \mathbb{P}(A_2 \cap A_3) - 2 \cdot \mathbb{P}(B_0).$$

Applying independence, we compute the probability of double intersection, and get:

$$\mathbb{P}(C) = 0.2 \cdot 0.4 + 0.2 \cdot 0.5 + 0.4 \cdot 0.5 - 2 \cdot 0.04 = 30\%.$$

3. (10 points) An actuary has discovered that policyholders are three times as likely to file two claims as to file four claims. The number of claims filed has a Poisson distribution. Calculate the variance of the number of claims filed.

Solution: We model $X \sim \text{Poi}(\lambda)$ and get:

$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \ldots$$

We have the condition $\mathbb{P}(X = 2) = 3 \cdot \mathbb{P}(X = 4)$ which is equivalent to

$$\frac{\lambda^2}{2!} = 3 \cdot \frac{\lambda^4}{4!} \Rightarrow \lambda^2 = 4 \Rightarrow \lambda = 2.$$

The variance of X is $\lambda = 2$.

4. Sometimes the stock market fluctuations X are modeled by the generalized Laplace distribution with MGF for some $a, b > 0$:

$$M(t) = \mathbb{E}\left[e^{tX}\right] = (1 - at^2)^{-b}$$

(a) (10 points) Find the variance of these fluctuations.

(b) (5 points) Find the expected returns $\mathbb{E}[e^X]$.

Solution: (a) Take two derivatives:

$$M'(t) = 2abt(1 - at^2)^{-b-1};$$
$$M''(t) = 2ab(1 - at^2)^{-b-1} + 2a^2b(b+1)(1 - at^2)^{-b-2}.$$

Plugging in $t = 0$, we have: The first moment (the mean) and the second moment are

$$\mathbb{E}[X] = M'(0) = 0;$$
$$\mathbb{E}[X^2] = M''(0) = 2ab.$$

Thus, the variance is $\mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 2ab$.

(b) Simply plug in $t = 1$: $\mathbb{E}[e^X] = M(1) = (1 - a)^{-b}$.

5. Consider independent identically distributed random variables $X_1, \ldots, X_{100}$, each with density $f(x) = 3x^2$ for $0 \le x \le 1$.

   (a) (10 points) Approximate S as a normal random variable.

   (b) (5 points) Find $\mathbb{P}(74 \le S \le 77)$.

   (c) (5 points) Find the u such that $\mathbb{P}(S \ge u) = 99\%$.

Solution: (a) The first and second moments of X_k are

$$\mu = \mathbb{E}[X_k] = \int_0^1 x f(x)\,\mathrm{d}x = \int_0^1 (3x^3)\,\mathrm{d}x = \frac{3x^4}{4}\Big|_{x=0}^{x=1} = \frac{3}{4} = 0.75;$$
$$\mathbb{E}[X_k^2] = \int_0^1 x^2 f(x)\,\mathrm{d}x = \int_0^1 (3x^4)\,\mathrm{d}x = \frac{3x^5}{5}\Big|_{x=0}^{x=1} = \frac{3}{5} = 0.6.$$

Therefore, the variance of X_k is $\sigma^2 = 0.6 - 0.75^2 = 0.0375$. By independence, we have:

$$\mathbb{E}[S] = \mu + \ldots + \mu = 100\mu = 75,$$
$$\mathrm{Var}(S) = \sigma^2 + \ldots + \sigma^2 = 100\sigma^2 = 3.75.$$

Thus we can model $S \sim \mathcal{N}(75, 3.75)$. We can represent $S = 75 + \sqrt{3.75} \cdot Z$ for $Z \sim \mathcal{N}(0,1)$.

(b) Then $S \le 77$ is equivalent to $Z \le 2/\sqrt{3.75} = 1.03$. This has probability 0.86. And $S \le 74$ is equivalent to $S \le -1/\sqrt{3.75} = -0.51$. This has probability $1 - 0.6950 = 0.305$. The difference is

$$\mathbb{P}(74 \le S \le 77) = 0.86 - 0.305 = 0.555.$$

(c) With probability 99%, $Z > -2.326$, and $S > 75 + \sqrt{3.75} \cdot (-2.326) = 70.5$. Thus $u = 70.5$.

6. (20 points) A company takes out an insurance policy to cover accidents. Each month is either good (no accidents) or bad (at least one accident). Various months are good or bad independently of each other. The probability that any given month is good is p. Find the probability that there are at least 3 good months before the second bad month.

Solution: Find the complementary probability that there are at most two good months before the second bad month. This can happen in two cases (with $q = 1 - p$, the probability of any given month is bad):

- Bad, Bad: Probability q^2.
- Good, Bad, Bad: Probability pq^2.
- Bad, Good, Bad: Probability pq^2.
- Good, Good, Bad, Bad: Probability p^2q^2.
- Good, Bad, Good, Bad: Probability p^2q^2.
- Bad, Good, Good, Bad: Probability p^2q^2.

Therefore, the overall probability is $q^2 + 2pq^2 + 3p^2q^2$. The answer is the complementary probability, which is $1 - (q^2 + 2pq^2 + 3p^2q^2)$.

7. US stocks in % have returns L with mean 7 and standard deviation 13. International stocks in % have returns S with mean 9 and standard deviation 20. The correlation is 60%. We invest in a portfolio with weights 70/30. That is, portfolio returns are $P = 0.7L + 0.3S$.

   (a) (5 points) Find the mean of P.

   (b) (10 points) Find the standard deviation of P.

Solution: (a) We have $\mathbb{E}[P] = 0.7 \cdot \mathbb{E}[L] + 0.3 \cdot \mathbb{E}[S] = 0.7 \cdot 7 + 0.3 \cdot 9 = 8.6$.

(b) We compute: $\text{Cov}(L, S) = 13 \cdot 20 \cdot 0.6 = 156$. Also, $\text{Var}(L) = 13^2 = 169$ and $\text{Var}(S) = 20^2 = 400$. Thus

$$\begin{aligned}\text{Var}(P) &= \text{Var}(0.7L + 0.3S) = \text{Var}(0.7L) + 2\text{Cov}(0.7L, 0.3S) + \text{Var}(0.3S) \\ &= 0.7^2 \cdot \text{Var}(L) + 2 \cdot 0.7 \cdot 0.3 \cdot \text{Cov}(L, S) + 0.3^2 \cdot \text{Var}(S) \\ &= 0.7^2 \cdot 169 + 2 \cdot 0.7 \cdot 0.3 \cdot 156 + 0.3^2 \cdot 400 = 184.33.\end{aligned}$$

The standard deviation is therefore $\sqrt{184.33} = 13.6$.

8. (20 points) Pick uniformly a random point on a unit disc $\{(x, y) \mid x^2 + y^2 \leq 1\}$. Find the density of the radius $r = \sqrt{x^2 + y^2}$.

Solution: Let us find the probability that $r \leq c$, where $c \in (0, 1)$. This is the probability of getting in the disc $D(c)$ of radius c. This is equal to the ratio

$$F(c) = \mathbb{P}(r \leq c) = \frac{\text{area}(D(c))}{\text{area}(D(1))} = \frac{\pi c^2}{\pi} = c^2.$$

Here, F is the cumulative distribution function. The density is $F'(c) = 2c$.

Cumulative Probabilities of the Standard Normal Distribution

The table gives the probabilities $\alpha = \Phi(z)$ to the left of given z–values for the standard normal distribution.

For example, the probability that a standard normal random variable Z is less than 1.53 is found at the intersection of the 1.5 rows and the 0.03 column, thus $\Phi(1.53) = P(Z \leq 1.53) = 0.9370$. Due to symmetry it holds $\Phi(-z) = 1 - \Phi(z)$ for all z.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Quantiles of the Standard Normal Distribution

For selected probabilities α, the table shows the values of the quantiles z_α such that $\Phi(z_\alpha) = P(Z \leq z_\alpha) = \alpha$, where Z is a standard normal random variable.

The quantiles satisfy the relation $z_{1-\alpha} = -z_\alpha$.

α	0.9	0.95	0.975	0.99	0.995	0.999
z_α	1.282	1.645	1.960	2.326	2.576	3.090