Fall Semester 2019
Stat 461 Final Exam
Friday, December 13

Name _____ Student ID #_____

Your exam should consist of 10 problems, followed by a Z-table.

You are allowed a double-sided handwritten note and a calculator.

Problem	Points	S C O R E
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (10 points) Find the cumulative distribution function for X with probability density function

$$f(x) = \begin{cases} 3x^{-4}, \, x \geq 1; \\ 0, \quad \text{else.} \end{cases}$$

Solution: We have $F(x) = 0$ for $x \leq 1$, because $X \geq 1$. For $x \geq 1$, we get:

$$F(x) = \int_1^x f(y) \, \mathrm{d}y = \int_1^x (3y^{-4}) \, \mathrm{d}y = y^{-3}\Big|_{y=1}^{y=x} = 1 - x^{-3}.$$

Thus the answer is

$$F(x) = \boxed{\begin{cases} 1 - x^{-3}, \, x \geq 1; \\ 0, \, x \leq 1. \end{cases}}$$

2. (10 points) Insurance claims arrive with intensity 0.5: That is, interarrival times are i.i.d. exponential random variables with $\lambda = 0.5$. Let X be the time when the third claim arrives. Using moment generating functions, find $\mathbb{E}[X^3]$ and $\mathbb{E}\left[e^{X/3}\right]$.

Solution: The time between $k-1$ and kth claims is $X_k \sim \text{Exp}(\lambda)$ with $\mathbb{E}[X_k] = \lambda^{-1} = 2$, thus $\lambda = 0.5$. We have: $X = X_1 + X_2 + X_3$, thus $X \sim \Gamma(3, 0.5)$. The MGF of X is

$$M(t) = \mathbb{E}\left[e^{tX}\right] = \left(\frac{0.5}{0.5 - t}\right)^3 = (1 - 2t)^{-3}.$$

Thus $\mathbb{E}\left[e^{X/3}\right] = M(1/3) = \boxed{27}$. Next, $\mathbb{E}[X^3] = M^{(3)}(0)$. Compute the third derivative:

$$M^{(3)}(t) = (-2)(-3)(-2)(-4)(-2)(-5)(1 - 2t)^{-6} \;\Rightarrow\; M^{(3)}(0) = \boxed{480}$$

3. (10 points) A financial analyst modeled 10-year total returns on utilities stocks as $U \sim \mathcal{N}(2, 1.44)$ (low risk, low reward), and 10-year total returns on health care stocks as $H \sim \mathcal{N}(5, 9)$ (high risk, high reward). Correlation between these random variables is assumed to be 0.2. This investor distributes 20% of capital into utilities stocks, 60% of capital into health care stocks, and 20% of capital into 10-year certificate of deposit which has fixed 10-year total return 0.45 (no risk, very low return). Thus the return of this portfolio is

$$R := 0.2U + 0.6H + 0.2 \cdot 0.45.$$

Find the probability that this return is negative, assuming that R has normal distribution too.

Solution: This random variable R has mean

$$\mathbb{E}[R] = 0.2 \cdot \mathbb{E}[U] + 0.6 \cdot \mathbb{E}[H] + 0.2 \cdot 0.45 = 0.2 \cdot 2 + 0.6 \cdot 5 + 0.2 \cdot 0.45 = 3.49,$$

Covariance of H and U is found as

$$\mathrm{Cov}(U, R) = \mathrm{stdev}(U)\mathrm{stdev}(R)\mathrm{corr}(U, R) = \sqrt{1.44} \cdot \sqrt{9} \cdot 0.2 = 1.2 \cdot 3 \cdot 0.2 = 0.72.$$

Thus the variance of R can be found as

$$\mathrm{Var}(R) = 0.2^2 \cdot \mathrm{Var}(U) + 0.6^2 \cdot \mathrm{Var}(H) + 2 \cdot 0.2 \cdot 0.6 \cdot \mathrm{Cov}(U, H) = 0.2^2 \cdot 1.44 + 0.6^2 \cdot 9 + 2 \cdot 0.2 \cdot 0.6 \cdot 0.72 = 3.4704.$$

We can represent $R = 3.49 + \sqrt{3.4704} \cdot Z$ for $Z \sim \mathcal{N}(0, 1)$. Thus $R < 0$ is equivalent to

$$Z < -\frac{3.49}{\sqrt{3.4704}} = -1.87,$$

which has probability $\mathbb{P}(Z < -1.87) = \mathbb{P}(Z > 1.87) = 1 - 0.9693 = \boxed{0.0307}$

4. (10 points) An insurance company has $N = 1000$ clients, each has an accident with loss distributed as Exp(2). All losses are independent. Each client is charged the same premium x. Using the Central Limit Theorem, find an x such that the company can pay all its claims with probability 95%.

Solution: Let $X_i \sim \text{Exp}(2)$ be the loss of the ith client, $i = 1, \ldots, N$. Then $S = X_1 + \ldots + X_N$ is the total loss, which the company needs to cover, and Nx is the total amount of cash the company has. We wish to find x such that $\mathbb{P}(S \leq Nx) = 90\%$. By the Central Limit Theorem,

$$\frac{S - \mathbb{E}S}{\sqrt{\text{Var}\, S}} \sim \mathcal{N}(0, 1).$$

By properties of the exponential distribution, $\mathbb{E}[X_i] = 0.5$ and $\text{Var}(X_i) = 0.25$. Thus

$$\mathbb{E}[S] = \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_N] = 1000 \cdot 0.5 = 500,$$

and by independence of $X_1, \ldots, X_N$, we get:

$$\text{Var}(S) = \text{Var}(X_1) + \ldots + \text{Var}(X_N) = 1000 \cdot 0.25 = 250.$$

From the Z-table, we get: With probability 95%,

$$\frac{S - 500}{\sqrt{250}} \leq 1.645 \;\Leftrightarrow\; S \leq 500 + \sqrt{250} \cdot 1.645 = 526.$$

Thus we need $Nx = 520.27 \;\Rightarrow x = \boxed{0.526}$

5. (10 points) In the population, 10% of people have a germ. A medical test identifies this germ if the patient has it with probability 80%, and it gives correct result if the patient does not have this germ with probability 90%. Given that a randomly chosen person has positive result, what is the probability that he, in fact, has this germ?

Solution: Let F_1 and F_2 be the events that this patient has a germ and does not have a germ. Let A be the event that the result is positive. Then

$$\mathbb{P}(F_1) = 0.1,\ \mathbb{P}(F_2) = 0.9,\ \mathbb{P}(A \mid F_1) = 0.8,\ \mathbb{P}(A \mid F_2) = 0.1.$$

By Bayes' formula,

$$\mathbb{P}(F_1 \mid A) = \frac{\mathbb{P}(A \mid F_1) \cdot \mathbb{P}(F_1)}{\mathbb{P}(A \mid F_1) \cdot \mathbb{P}(F_1) + \mathbb{P}(A \mid F_2) \cdot \mathbb{P}(F_2)} = \frac{0.1 \cdot 0.8}{0.1 \cdot 0.8 + 0.9 \cdot 0.1} = \boxed{\frac{8}{17}}$$

6. (10 points) An investor has 8 assets, each loses money with probability 20%, independently of others. Find the probability that there are at most two losses.

Solution: The number of losses is $X \sim \text{Bin}(8, 0.2)$. Thus

$$\mathbb{P}(X = k) = \binom{8}{k} \cdot 0.2^k \cdot 0.8^{8-k}, \quad k = 0, \ldots, 8.$$

In particular, the probability $\mathbb{P}(X \leq 2)$ is equal to

$$\binom{8}{0} \cdot 0.8^8 + \binom{8}{1} \cdot 0.2 \cdot 0.8^7 + \binom{8}{2} \cdot 0.2^2 \cdot 0.8^6 = 0.1678 + 0.3355 + 0.2936 = \boxed{0.797}$$

7. (10 points) Take two random variables X and Y with joint density

$$f(x,y) = \begin{cases} x+y, \ 0 \le x \le 1, \, 0 \le y \le 1, \\ 0, \quad \text{else.} \end{cases}$$

Find the conditional density and expectation of X given $Y = 1$.

Solution: The marginal density of Y is given by

$$f_Y(y) = \int_0^1 f(x,y)\,\mathrm{d}x = \frac{1}{2} + y, \ 0 \le y \le 1.$$

The conditional density of X given Y is given by

$$f(x \mid y) = \frac{f(x,y)}{f_Y(y)} = \frac{x+y}{y+1/2}, \ 0 \le x \le 1.$$

For $Y = 1$, this conditional density becomes

$$f(x \mid y = 1) = \frac{x+1}{1/2+1} = \boxed{\frac{3}{2}x + \frac{3}{2}, \ 0 \le x \le 1}$$

The conditional expectation is equal to

$$\mathbb{E}[X \mid Y = 1] = \int_0^1 x f(x \mid y = 1)\,\mathrm{d}x = \frac{3}{2}\int_0^1 x(x+1)\,\mathrm{d}x = \frac{3}{2}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_{x=0}^{x=1} = \frac{3}{2}\cdot\frac{5}{6} = \boxed{\frac{5}{9}}$$

8. (10 points) Take two independent random variables $X \sim \text{Exp}(4)$ and $Y \sim \text{Exp}(3)$. Find the density and the mean value of $\min(X, Y)$.

Solution: For every $u \geq 0$, we get:

$$\mathbb{P}(\min(X, Y) > u) = \mathbb{P}(X > u\, Y > u) = \mathbb{P}(X > u) \cdot \mathbb{P}(Y > u).$$

We compute the tail functions for X and Y: $\mathbb{P}(X > u) = e^{-4u}$ and $\mathbb{P}(Y > u) = e^{-3u}$ for $u \geq 0$. Therefore, $\mathbb{P}(\min(X, Y) > u) = e^{-4u} \cdot e^{-3u} = e^{-7u}$ for $u \geq 0$. Thus the CDF is $F(u) = 1 - e^{-7u}$ for $u \geq 0$; and $F(u) = 0$ for $u < 0$. The density is $F'(u) = \boxed{7e^{-7u},\ u \geq 0}$ This is Exp(7), with mean $\boxed{1/7}$

9. (10 points) Take two events A and B, each having probability 0.48. The probability that neither A nor B happens is 0.1. Find $\mathbb{P}(A \mid B)$.

Solution: The probability of $A \cup B$ is $1 - 0.1 = 0.9$. By the inclusion-exclusion formula,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = 0.48 + 0.48 - 0.9 = 0.06.$$

Thus the conditional probability is given by

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{0.06}{0.48} = \frac{1}{8} = \boxed{0.125}$$

10. (10 points) Assume each Fortune 500 company defaults with probability 0.5%, independently of others. Using the Poisson approximation, find the probability that there is at most one default.

Solution: We have: $X \sim \text{Poi}(\lambda)$ for $\lambda = 500 \cdot 0.5\% = 2.5$. Thus

$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \ k = 0, 1, 2, \ldots$$

$$\mathbb{P}(X \leq 1) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1) = e^{-2.5} + 2.5e^{-2.5} = 3.5e^{-2.5} = \boxed{0.287}$$

Cumulative Probabilities of the Standard Normal Distribution

The table gives the probabilities $\alpha = \Phi(z)$ to the left of given z–values for the standard normal distribution.

For example, the probability that a standard normal random variable Z is less than 1.53 is found at the intersection of the 1.5 rows and the 0.03 column, thus $\Phi(1.53) = P(Z \leq 1.53) = 0.9370$. Due to symmetry it holds $\Phi(-z) = 1 - \Phi(z)$ for all z.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Quantiles of the Standard Normal Distribution

For selected probabilities α, the table shows the values of the quantiles z_α such that $\Phi(z_\alpha) = P(Z \leq z_\alpha) = \alpha$, where Z is a standard normal random variable.

The quantiles satisfy the relation $z_{1-\alpha} = -z_\alpha$.

α	0.9	0.95	0.975	0.99	0.995	0.999
z_α	1.282	1.645	1.960	2.326	2.576	3.090