Stat 461/661 Final Exam
Wednesday, December 19

Name _____ Student ID #_____

Your exam should consist of 10 problems, followed by a Z-table. Check!

You are allowed a double-sided handwritten note and a calculator.

Problem	Points	S C O R E
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (10 points) Based on pre-trial speculation, the probability that a jury returns a guilty verdict in a certain high-profile murder case is thought to be 15% if the defense can discredit the police department, and 80% if they cannot. Veteran court observers believe that the skilled defense attorneys have a 70% chance of convincing the jury that the police either contaminated or planted some evidence. What is the probabiliy that the jury returns a guilty verdict?

   Solution: We have $F_1 =$ can discredit, $F_2 =$ cannot discredit, $A =$ guilty verdict, with

   $$\mathbb{P}(F_1) = 0.7,\ \mathbb{P}(F_2) = 0.3,\ \mathbb{P}(A \mid F_1) = 0.15,\ \mathbb{P}(A \mid F_2) = 0.8.$$

   Thus we get:

   $$\mathbb{P}(A) = \mathbb{P}(A \mid F_1) \cdot \mathbb{P}(F_1) + \mathbb{P}(A \mid F_2) \cdot \mathbb{P}(F_2) = 0.7 \cdot 0.15 + 0.3 \cdot 0.8 = \boxed{0.345}$$

2. (10 points) An IT center uses 9 aging disk drives for storage. The probability that any one of them is out of service is 0.06. For the center to function properly, at least 7 drives must be available. What is the probability of that?

Solution: The number X of working disks has a Binomial distribution $\text{Bin}(9, 0.94)$. Thus

$$\mathbb{P}(X = k) = \binom{9}{k} 0.94^k 0.06^{9-k}, \ k = 0, \ldots, 9.$$

From here, we get:

$$\mathbb{P}(X \geq 7) = \binom{9}{9} 0.94^9 + \binom{9}{8} 0.94^8 0.06 + \binom{9}{7} 0.94^7 0.06^2 = \boxed{0.986}$$

3. (10 points) Take two independent random variables X, Y, each of which takes values $0, +1, -1$ with probabilities $0.4, 0.3, 0.3$. Find the distribution, expectation, variance of $Z = X - 2Y + 1$.

Solution: We calculate expectation and variance of X and Y:

$$\begin{aligned}
\mathbb{E}[X] = \mathbb{E}[Y] &= 0.4 \cdot 0 + 0.3 \cdot 1 + 0.3 \cdot (-1) = 0, \\
\mathbb{E}\left[X^2\right] &= 0.4 \cdot 0^2 + 0.3 \cdot 1^2 + 0.3 \cdot (-1)^2 = 0.6, \\
\mathrm{Var}(Y) = \mathrm{Var}(X) &= \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2 = 0.6.
\end{aligned}$$

By independence of X and Y,

$$\mathbb{E}[Z] = \mathbb{E}[X] - 2\mathbb{E}[Y] + 1 = \boxed{1} \quad \mathrm{Var}(Z) = \mathrm{Var}(X) + 4\,\mathrm{Var}(Y) = 0.6 + 4 \cdot 0.6 = \boxed{3}$$

Z/P	-1	0	1
-1	2/0.09	0/0.12	-2/0.09
0	3/0.12	1/0.16	-1/0.12
1	4/0.09	2/0.12	0/0.09

Thus distribution of Z is given by

Z	-2	-1	0	1	2	3	4
P	0.09	0.12	0.21	0.16	0.21	0.12	0.09

4. (10 points) A continuous random variable X has cumulative distribution function

$$F_X(x) = \begin{cases} 0, \ x \leq 0; \\ \frac{1}{12}(x^2 + x^3), \ 0 \leq x \leq 2; \\ 1, \ x \geq 2. \end{cases}$$

Find the density (probability density function) of X, and $\mathbb{E}X$.

Solution: Density is given by

$$f(x) = F_X'(x) = \frac{1}{6}x + \frac{1}{4}x^2, \ 0 \leq x \leq 2,$$

and zero elsewhere. Thus expectation is given by

$$\mathbb{E}[X] = \int_0^2 x f(x)\,\mathrm{d}x = \int_0^2 \left[\frac{1}{6}x^2 + \frac{1}{4}x^3\right]\,\mathrm{d}x = \left[\frac{1}{18}x^3 + \frac{1}{16}x^4\right]\Big|_{x=0}^{x=2} = \boxed{\frac{13}{9}}$$

5. (10 points) A financial analyst models 10-year return of a portfolio as $R = e^X$ for $X \sim \mathcal{N}(2.3, 9)$. Find the density of R, and the probability that this return is greater than 24.

Solution: We can represent $X = 2.3 + 3Z$ for $Z \sim \mathcal{N}(0, 1)$. Thus

$$R > 24 \;\Leftrightarrow\; 2.3 + 3Z > \ln(24) \Leftrightarrow\; Z > \frac{1}{3}\left(\ln(24) - 2.3\right) = 0.29.$$

The probability of this is (from the Z-table) is $1 - 0.6141 = \boxed{0.3859}$ Next, the density of X is

$$f(x) = \frac{1}{3\sqrt{2\pi}} \exp\left[-\frac{(x-2.3)^2}{18}\right].$$

For any $0 < a < b$, we have:

$$\mathbb{P}(a < R < b) = \mathbb{P}(\ln(a) < X < \ln(b)) = \int_{\ln(a)}^{\ln(b)} f(x)\,\mathrm{d}x = \int_a^b f(\ln y)\,\mathrm{d}\ln y = \int_a^b f(\ln y)\frac{\mathrm{d}y}{y}.$$

We changed variables $x = ln(y)$. Thus the density of R is

$$\frac{1}{y} f(\ln y) = \frac{1}{3\sqrt{2\pi}y} \exp\left[-\frac{(\ln y - 2.3)^2}{18}\right].$$

6. (10 points) Using the monent generating function, compute $\mathbb{E}[X^4]$ for $X \sim \mathcal{N}(0,1)$.

Solution: The MGF of X is $M(x) = e^{t^2/2}$, and

$$M^{(4)}(t) = e^{t^2/2}(t^4 + 6t^2 + 3).$$

Thus the fourth moment is equal to

$$\mathbb{E}\left[X^4\right] = M^{(4)}(0) = \boxed{3}$$

7. (10 points) An insurance company has three types of clients: high-risk, medium-risk, and low-risk, 100 clients of each type. For each client, expenses are modeled as a random variable with

Client	Mean	Variance
Low-risk	2.3	0.2
Medium-risk	3.2	0.4
High-risk	4	0.7

Find the value-at-risk for confidence level 99%.

Solution: Total expenses are $S = X_1 + \ldots + X_{100} + Y_1 + \ldots + Y_{100} + Z_1 + Z_{100}$, where X_i, Y_i, Z_i are expenses for low-risk, medium-risk, and high-risk clients. Thus

$$\mathbb{E}[S] = 2.3 \cdot 100 + 3.2 \cdot 100 + 4 \cdot 100 = 950,$$
$$\mathrm{Var}[S] = 0.2 \cdot 100 + 0.4 \cdot 100 + 0.7 \cdot 100 = 130.$$

By Central Limit Theorem,

$$\frac{S - 950}{\sqrt{130}} \approx \mathcal{N}(0, 1).$$

Thus value-at-risk is $950 + \sqrt{130} \cdot 2.326 = \boxed{976.52}$

8. (10 points) For i.i.d. $X_1, X_2, \ldots$ distributed as Uni$[0, 2]$, define

$$M_n := \frac{1}{n} [X_1 + \ldots + X_n] .$$

(STAT 461) Using Chebyshev's inequality, estimate the following probability from above:

$$\mathbb{P} (|M_{100} - 1| \geq 0.5) .$$

(STAT 661) For an $\varepsilon > 0$, find all n such that from Chebyshev's inequality we get the estimate:

$$\mathbb{P} (|M_n - 1| \geq \varepsilon) \leq 0.1.$$

Solution: Expectation of M_n is equal to 1, and variance is

$$\mathrm{Var}(M_n) = \frac{1}{n^2} [\mathrm{Var}(X_1) + \ldots + \mathrm{Var}(X_n)] = \frac{1}{3n},$$

because variance of Uni$[0, 2]$ is $1/3$ (easy to compute). By Chebyshev's inequality,

$$\mathbb{P} (|M_n - 1| \geq \varepsilon) \leq \frac{\mathrm{Var}(M_n)}{\varepsilon^2} = \frac{1}{3n\varepsilon^2}.$$

For $n = 100$ and $\varepsilon = 0.5$, we get

$$\frac{1}{3n\varepsilon^2} = \boxed{\frac{1}{75}}$$

In general, for fixed $\varepsilon > 0$ we have:

$$\frac{1}{3n\varepsilon^2} \leq 0.1 \ \Leftrightarrow n \geq \boxed{0.3\varepsilon^2}$$

9. (10 points) Find $\mathbb{P}(X \geq 2 \mid X \leq 4)$ for a geometric random variable X with $p = 0.65$.

Solution: $\mathbb{P}(X = k) = p(k) = 0.65 \cdot 0.35^{k-1},\ k = 1, 2, \ldots$ In particular,

$$p(1) = 0.65,\ p(2) = 0.2275,\ p(3) = 0.0796,\ p(4) = 0.0279.$$

$$\mathbb{P}(X \geq 2 \mid X \leq 4) = \frac{\mathbb{P}(2 \leq X \leq 4)}{\mathbb{P}(X \leq 4)} = \frac{p(2) + p(3) + p(4)}{p(1) + p(2) + p(3) + p(4)} = \boxed{0.34}$$

10. (10 points) Consider two random variables (X, Y) with joint density

$$f(x, y) = 2e^{-x-y},\ 0 < x < y,$$

and zero elsewhere. Find the conditional density of X given $Y = y$.

Solution: Marginal density of Y is

$$f_Y(y) = \int_0^y f(x, y)\,\mathrm{d}x = \left(-2e^{-x-y}\right)\Big|_{x=0}^{x=y} = 2e^{-y} - 2e^{-2y}.$$

Thus conditional density is

$$\frac{f(x, y)}{f_Y(y)} = \frac{2e^{-x-y}}{2e^{-y} - 2e^{-2y}} = \boxed{\frac{e^{-x}}{1 - e^{-y}},\ 0 \le x \le y}$$

Cumulative Probabilities of the Standard Normal Distribution

The table gives the probabilities $\alpha = \Phi(z)$ to the left of given z–values for the standard normal distribution.

For example, the probability that a standard normal random variable Z is less than 1.53 is found at the intersection of the 1.5 rows and the 0.03 column, thus $\Phi(1.53) = P(Z \leq 1.53) = 0.9370$. Due to symmetry it holds $\Phi(-z) = 1 - \Phi(z)$ for all z.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Quantiles of the Standard Normal Distribution

For selected probabilities α, the table shows the values of the quantiles z_α such that $\Phi(z_\alpha) = P(Z \leq z_\alpha) = \alpha$, where Z is a standard normal random variable.

The quantiles satisfy the relation $z_{1-\alpha} = -z_\alpha$.

α	0.9	0.95	0.975	0.99	0.995	0.999
z_α	1.282	1.645	1.960	2.326	2.576	3.090